2-6 The Quadratic Formula

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I CAN… solve quadratic equations using the Quadratic Formula.

VOCABULARY

- discriminant
- Quadratic Formula

MAFS.912.A-REI.2.4.b–Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.
Also N-CN.3.7, A-REI.2.4, A-REI.2.4.a
MAFS.K12.MP.3.1, MP.5.1, MP.7.1

Activity Assess

EXPLORE & REASON

You can complete the square to solve the general quadratic equation, $ax^2 + bx + c = 0$.

A. Construct Arguments
Justify each step in this general solution.

B. What must be true of the value of $b^2 - 4ac$ if the equation $ax^2 + bx + c = 0$ has two non-real solutions? If it has just one solution?

$$ax^2 + bx + c = 0$$
$$ax^2 + bx = -c$$
$$x^2 + \left(\frac{b}{a}\right)x = -\frac{c}{a}$$
$$x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$
$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

ESSENTIAL QUESTION

How can you use the Quadratic Formula to solve quadratic equations or to predict the nature of their solutions?

EXAMPLE 1 Solve Quadratic Equations

What are the solutions to the equation?

A. $3x^2 - 4x - 9 = 0$

The **Quadratic Formula**, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, provides the solutions of the quadratic equation $ax^2 + bx + c = 0$, for $a \neq 0$.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ — Write the Quadratic Formula.

$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-9)}}{2(3)}$ — Substitute 3 for a, -4 for b, and -9 for c.

$= \frac{4 \pm \sqrt{124}}{6}$ — Simplify.

$= \frac{4 \pm 2\sqrt{31}}{6}$

$= \frac{2 \pm \sqrt{31}}{3}$

The solutions are

$x = \frac{2 + \sqrt{31}}{3}$ and $x = \frac{2 - \sqrt{31}}{3}$.

CONTINUED ON THE NEXT PAGE

USE APPROPRIATE TOOLS
The Quadratic Formula is a useful tool for finding solutions, particularly when an equation cannot be easily factored.

EXAMPLE 1 CONTINUED

B. How can you use the Quadratic Formula to solve $x^2 - 9x + 27 = 0$?

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ ········ Write the Quadratic Formula.

$= \frac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(27)}}{2(1)}$ ········ Substitute 1 for a, -9 for b, and 27 for c.

$= \frac{9 \pm \sqrt{-27}}{2}$ ········ Simplify.

$= \frac{9 \pm i\sqrt{27}}{2}$ ········ $\sqrt{-1} = i$

$= \frac{9 \pm 3i\sqrt{3}}{2}$ ········ Simplify.

The solutions are $x = \frac{9 + 3i\sqrt{3}}{2}$ and $x = \frac{9 - 3i\sqrt{3}}{2}$.

GENERALIZE
Look for relationships between the coefficients of a quadratic equation and its solutions. If $a = 1$, then the sum of the solutions is the opposite of the x-coefficient, b, and their product is the constant coefficient, c.

Try It! **1.** Solve using the Quadratic Formula.

a. $2x^2 + 6x + 3 = 0$ **b.** $3x^2 - 2x + 7 = 0$

EXAMPLE 2 Choose a Solution Method

Solve the equation $6x^2 - 7x - 20 = 0$ using two different methods. Which do you prefer and why?

STUDY TIP
When you substitute a negative number into a formula, such as the Quadratic Formula, use parentheses to help keep track of the effect of the sign.

Using the Quadratic Formula:

Let $a = 6$, $b = -7$, and $c = -20$

$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(6)(-20)}}{2(6)}$

$= \frac{7 \pm \sqrt{49 + 480}}{12}$

$= \frac{7 \pm \sqrt{529}}{12}$

$= \frac{7 \pm 23}{12}$

$x = \frac{7 + 23}{12} = \frac{30}{12} = \frac{5}{2}$, and

$x = \frac{7 - 23}{12} = -\frac{16}{12} = -\frac{4}{3}$

Factoring by Grouping:

$6x^2 - 7x - 20 = 0$

$6x^2 - 15x + 8x - 20 = 0$

$3x(2x - 5) + 4(2x - 5) = 0$

$(3x + 4)(2x - 5) = 0$

$x = -\frac{4}{3}$ and $x = \frac{5}{2}$

You may also find the factorization through trial and error.

Both solution methods give the same result. Factoring may be more efficient, but the Quadratic Formula *always works*, regardless of whether the function has real or imaginary roots.

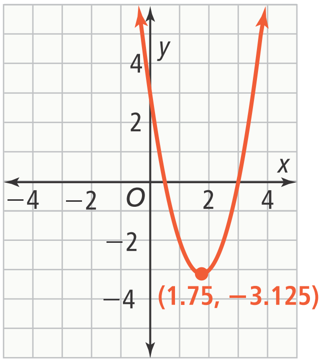
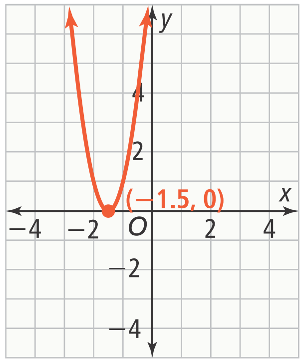
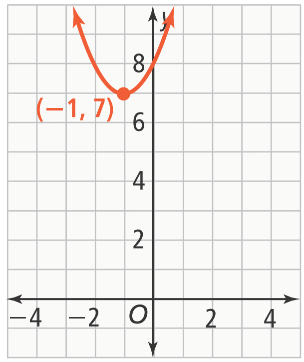
Try It! **2.** Solve the equation $6x^2 + x - 15 = 0$ using the Quadratic Formula and another method.

CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Identify the Number of Real-Number Solutions

How can you determine the number and type of roots for a quadratic equation?

Graph each equation. Then use the quadratic formula to find the roots.

$y = 2x^2 - 7x + 3$	$y = 4x^2 + 12x + 9$	$y = x^2 + 2x + 8$
		
$2x^2 - 7x + 3 = 0$	$4x^2 + 12x + 9 = 0$	$x^2 + 2x + 8 = 0$
$x = \frac{7 \pm \sqrt{(-7)^2 - 4(2)(3)}}{2(2)}$ $x = \frac{7 \pm \sqrt{25}}{4}$	$x = \frac{-12 \pm \sqrt{(12)^2 - 4(4)(9)}}{2(4)}$ $x = \frac{-12 \pm \sqrt{0}}{8}$	$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(8)}}{2(1)}$ $x = \frac{-2 \pm \sqrt{-28}}{2}$
$x = \frac{7 \pm 5}{4}$	$x = \frac{-12}{8}$	$x = -1 \pm i\sqrt{7}$
The radicand is 25. Adding and subtracting $\sqrt{25}$ gives two different roots.	The radicand is 0. Adding and subtracting $\sqrt{0}$ gives one root.	The radicand is −28. Its square root is imaginary, so adding it and subtracting it will give two complex roots.
The graph shows two distinct *x*-intercepts and the quadratic formula shows two distinct real-number roots.	The graph shows one distinct *x*-intercept and the quadratic formula shows one real-number root.	The graph shows no *x*-intercepts and the quadratic formula shows two distinct complex-number roots.

The radicand in the quadratic formula is what determines the nature of the roots.

The **discriminant** of a quadratic equation in the form $ax^2 + bx + c = 0$ is the value of the radicand, $b^2 - 4ac$.

If $b^2 - 4ac > 0$, then $ax^2 + bx + c = 0$ has two real roots.

If $b^2 - 4ac = 0$, then $ax^2 + bx + c = 0$ has one real root.

If $b^2 - 4ac < 0$, then $ax^2 + bx + c = 0$ has two non-real roots.

Try It! **3.** Describe the nature of the solutions for each equation.

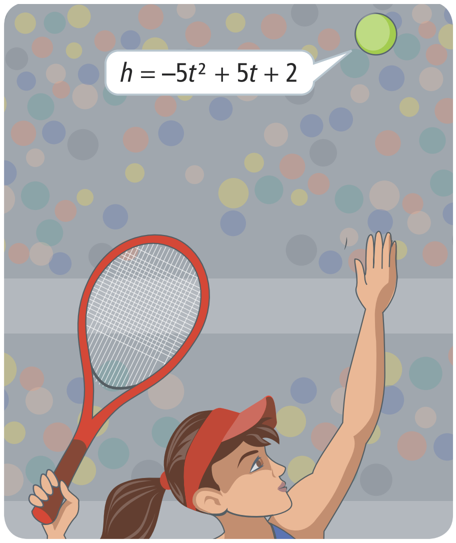
a. $16x^2 + 8x + 1 = 0$ **b.** $2x^2 - 5x + 6 = 0$

Activity Assess

APPLICATION

EXAMPLE 4 Interpret the Discriminant

Rachel is about to serve and tosses a tennis ball straight up into the air. The height, *h*, of the ball, in meters, at time *t*, in seconds is given by $h(t) = -5t^2 + 5t + 2$. Will the ball reach a height of 4 meters?



To see if $h = 4$ for some value of t, set the quadratic expression for h equal to 4, and solve.

$$-5t^2 + 5t + 2 = 4$$

Rewrite the equation in standard form:

$$-5t^2 + 5t - 2 = 0$$

$$a = -5,\ b = 5,\ c = -2$$

The discriminant is: $(5)^2 - 4(-5)(-2)$
$= 25 - 40$

$$25 - 40 = -15$$

$$-15 < 0$$

So the equation $h = 4$ does not have a real solution. Therefore, the ball does not reach 4 m.

Try It! **4.** According to the model of Rachel's serve, will the ball reach a height of 3 meters?

EXAMPLE 5 Use the Discriminant to Find a Particular Equation

What value(s) of *b* will cause $2x^2 + bx + 18 = 0$ to have one real solution?

For this equation, $a = 2$ and $c = 18$.

The equation will have a single rational solution when the discriminant is equal to 0.

$$b^2 - 4ac = 0$$

$$b^2 - 4(2)(18) = 0$$

$$b^2 - 144 = 0$$

$$b^2 = 144$$

$$b = \pm 12$$

There are two possible equations: $2x^2 + 12x + 18 = 0$ and $2x^2 - 12x + 18 = 0$.

STUDY TIP
Note that the equation $2x^2 + bx + 18 = 0$ will have two real solutions if $b > 12$ or $b < -12$. It will have two non-real solutions if $-12 < b < 12$.

Try It! **5.** Determine the value(s) of b that ensure $5x^2 + bx + 5 = 0$ has two non-real solutions.

Concept Summary | Assess

CONCEPT SUMMARY Key Features of the Quadratic Formula

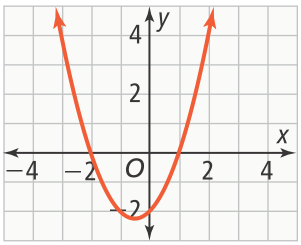
QUADRATIC FORMULA

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula is used to solve any quadratic equation: $ax^2 + bx + c = 0$, where $a \neq 0$.

USING THE DISCRIMINANT

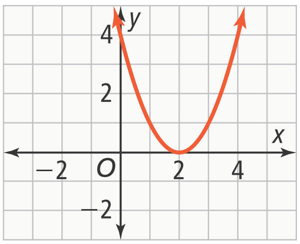
Predict the number and type of solutions using the discriminant, $b^2 - 4ac$.



$x^2 + x - 2 = 0$

$b^2 - 4ac > 0$

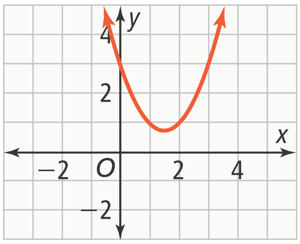
Two real solutions



$x^2 - 4x + 4 = 0$

$b^2 - 4ac = 0$

One real solution



$x^2 - 3x + 3 = 0$

$b^2 - 4ac < 0$

Two non-real solutions

Do You UNDERSTAND?

1. ESSENTIAL QUESTION How can you use the Quadratic Formula to solve quadratic equations or to predict the nature of their solutions?

2. **Vocabulary** Why is the discriminant a useful tool to use when solving quadratic equations?

3. **Error Analysis** Rick claims that the equation $x^2 + 5x + 9 = 0$ has no solution. Jenny claims that there are two solutions. Explain how Rick could be correct, and explain how Jenny could be correct.

4. **Use Appropriate Tools** What methods can you use to solve quadratic equations?

Do You KNOW HOW?

5. Describe the number and type of solutions of the equation $2x^2 + 7x + 11 = 0$.

6. Use the Quadratic Formula to solve the equation $x^2 + 6x - 10 = 0$.

7. At time t seconds, the height, h, of a ball thrown vertically upward is modeled by the equation $h = -5t^2 + 33t + 4$. About how long will it take for the ball to hit the ground?

8. Use the Quadratic Formula to solve the equation $x^2 - 8x + 16 = 0$. Is this the only way to solve this equation? Explain.

PRACTICE & PROBLEM SOLVING

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Practice Tutorial

Additional Exercises Available Online

UNDERSTAND

9. **Look for Relationships** How can you use the Quadratic Formula to factor a quadratic equation?

10. **Error Analysis** Describe and correct the error a student made in solving an equation.

$$x^2 - 5x + 5 = 0$$
$$a = 1,\ b = -5,\ c = 5$$
$$x = \frac{-5 \pm \sqrt{(-5)^2 - 4(1)(5)}}{2(1)}$$
$$= \frac{-5 \pm \sqrt{25 - 20}}{2}$$
$$= \frac{-5}{2} \pm \frac{\sqrt{5}}{2}$$

✗

11. **Mathematical Connections** What does the Quadratic Formula tell you about the graph of a quadratic function?

12. **Communicate Precisely** Explain your process for choosing a method for solving quadratic equations.

13. **Higher Order Thinking** Kelsey wants to use the Quadratic Formula to solve the equation $x^4 + 5x^2 - 5 = 0$. Is this possible? If so, describe the steps she should follow.

14. **Construct Arguments** Explain why the graph of the quadratic function $f(x) = x^2 + x + 5$ crosses the y-axis but does not cross the x-axis.

15. **Construct Arguments** Sage said that the Quadratic Formula does not always work. Sage used it to solve the equation $x^2 - 3x - 2 = -4$, with $a = 1$, $b = -3$, and $c = -2$. The formula gave $x = \frac{3 \pm \sqrt{17}}{2}$ as the solutions to the equation. When Sage checked, neither one of them satisfied the equation. How could you convince Sage that the Quadratic Formula does always work?

PRACTICE

Use the Quadratic Formula to solve each equation. SEE EXAMPLE 1

16. $x^2 - 10x + 25 = 0$

17. $x^2 + 2x + 2 = 0$

18. $5x^2 - 8x + 4 = 0$

19. $x^2 + 9x - 1 = 3x - 10$

20. $3x^2 - 20x - 7 = 0$

21. $-x^2 + 3x - 8 = 0$

Use the discriminant to identify the number and type of solutions for each equation. SEE EXAMPLE 3

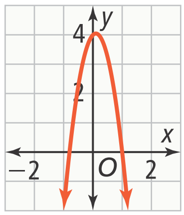
22. $25x^2 - 20x + 4 = 0$

23. $x^2 + 7x + 11 = 0$

24. $3x^2 - 8x - 10 = 0$

25. $2x^2 + 9x + 14 = 0$

Deon throws a ball into the air. The height, h, of the ball, in meters, at time t seconds is modeled by the function $h(t) = -5t^2 + t + 4$. SEE EXAMPLE 4



26. When will the ball hit the ground?

27. Will the ball reach a height of 5 meters?

Use any method to solve the equation. SEE EXAMPLE 2

28. $4x^2 + 7x - 11 = 0$

29. $x^2 + 4x + 4 = 100$

30. $3x^2 + x + 7 = x^2 + 10$

31. $6x^2 + 2x + 3 = 0$

Find the value(s) of k that will cause the equation to have the given number and type of solutions. SEE EXAMPLE 5

32. $5x^2 + kx + 5 = 0$, 1 real solution

33. $3x^2 + 12x + k = 0$, 2 real solutions

34. $kx^2 - 3x + 4 = 0$, 2 real solutions

PRACTICE & PROBLEM SOLVING

APPLY

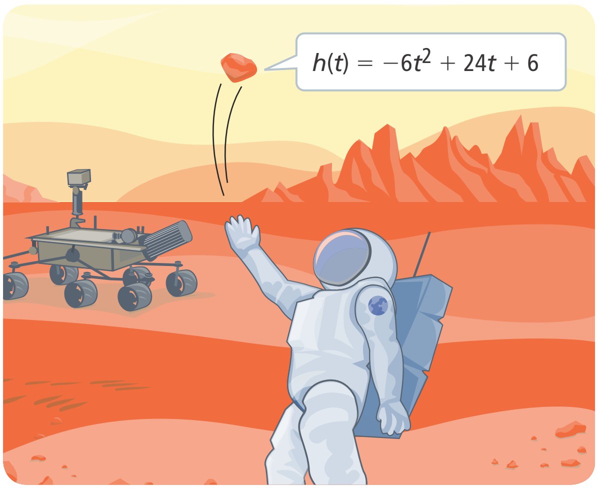
35. **Model With Mathematics** The table shows the average cost of tuition and fees at a public four-year college for an in-state student in recent years.

Academic Year	Tuition and Fees
2012–13	$9,006
2013–14	$9,077
2014–15	$9,161
2015–16	$9,410

a. Write an equation that can be used to find the average cost, C, of tuition after x years.

b. Use the model to predict when tuition will exceed $10,000.

36. **Make Sense and Persevere** The first astronaut on Mars tosses a rock straight up. The height, h, measured in feet after t seconds, is given by the function $h(t) = -6t^2 + 24t + 6$.



a. After how many seconds will the rock be 30 feet above the surface?

b. After how many seconds will the rock be 10 feet above the surface?

c. How many seconds will it take for the rock to return to the surface?

d. The same action on Earth is modeled by the equation $g(t) = -16t^2 + 24t + 6$. On Earth, how many seconds would it take for the rock to hit the ground?

ASSESSMENT PRACTICE

37. Select the equations that have two real solutions. A-REI.2.4.b

☐ **A.** $x^2 - 8x - 2 = 0$

☐ **B.** $2x^2 + 10x + 17 = 0$

☐ **C.** $4x^2 - 28x + 49 = 0$

☐ **D.** $x^2 + 10x - 25 = 4x + 2$

☐ **E.** $2x^2 + x + 10 = 5 - 4x - x^2$

38. **SAT/ACT** Which expression can be simplified to find the solution(s) of the equation $2x^2 - x - 15 = 0$?

Ⓐ $-1 \pm \dfrac{\sqrt{1 - 4(2)(-15)}}{2(2)}$

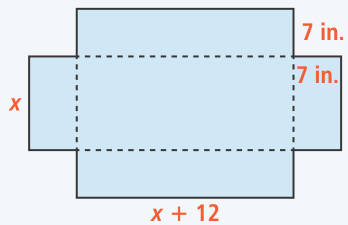
Ⓑ $\dfrac{1 \pm \sqrt{1 - 4(2)(-15)}}{2(2)}$

Ⓒ $\dfrac{1 \pm \sqrt{-1 - 4(2)(-15)}}{2(2)}$

Ⓓ $\dfrac{1 \pm \sqrt{1 - 4(2)(15)}}{2(2)}$

Ⓔ $\dfrac{1 \pm \sqrt{1 + 4(2)(-15)}}{2(2)}$

39. **Performance Task** Four congruent squares are cut from a rectangular piece of cardboard.



Part A. If the resulting flaps are folded up and taped together to make a box, write a function to represent the volume of the box in terms of the width of the original piece of cardboard.

Part B. What are the dimensions of the original cardboard, to the nearest tenth, if the volume of the box is 434 in.^{3}?