1-4 Arithmetic Sequences and Series

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I CAN… interpret arithmetic sequences and series.

VOCABULARY

- arithmetic sequence
- arithmetic series
- common difference
- explicit definition
- recursive definition
- sequence
- series
- sigma notation

MAFS.912.F-BF.1.2– Write arithmetic . . . sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.
Also F-BF.1.1, F-BF.1.1.a
MAFS.K12.MP.3.1, MP.4.1, MP.7.1

CONCEPTUAL UNDERSTANDING

COMMON ERROR
The common difference is always calculated by subtracting a term from the next term; $d = a_n - a_{n-1}$.

CRITIQUE & EXPLAIN

Yumiko and Hugo are looking at the table of data.

Input	Output
0	1
1	5
2	9
3	13
4	17

Yumiko writes $f(1) = 1 + 4 = 5$,

$f(2) = f(1) + 4 = 5 + 4 = 9$,

$f(3) = f(2) + 4 = 9 + 4 = 13$,

$f(4) = f(3) + 4 = 13 + 4 = 17$.

Hugo writes $g(x) = 1 + 4x$.

A. Describe the pattern Yumiko found for finding an output value.

B. Describe the pattern Hugo found for finding an output value.

C. **Use Structure** Compare the two methods. Which method would be more useful in finding the 100th number in the list? Why?

ESSENTIAL QUESTION

What is an arithmetic sequence, and how do you represent and find its terms and their sums?

EXAMPLE 1 Understand Arithmetic Sequences

A. Is the sequence arithmetic? If so, what is the common difference? What is the next term in the sequence?

3, 8, 13, 18, 23, …

This is a **sequence**, a function whose domain is the Natural numbers.

Create a table that shows the term number, or domain, and the term, or range.

Term Number	Term
1	3
2	8
3	13
4	18
5	23
6	?

+5, +5, +5, +5, +5

The difference between consecutive numbers in the range is 5.

An **arithmetic sequence** is a sequence with a constant difference between consecutive terms. This difference is known as the **common difference**, or *d*.

This sequence is an arithmetic sequence with the common difference, $d = 5$. The next term in the sequence is $23 + 5$, or 28.

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EXAMPLE 1 CONTINUED

B. How could you write a formula for finding the next term in the sequence?

STUDY TIP
An arithmetic sequence is a function, so you can write the terms using function notation.

Each term can be represented by $f(n)$ where n represents the number of the term.

So, for $n = 1$, $f(1) = 3$.

If $n > 1$, each term is the sum of the previous term and 5.

$$f(2) = f(1) + 5$$

$$f(3) = f(2) + 5$$

$$f(n) = f(n - 1) + 5$$

Write the general rule for an arithmetic sequence as a piecewise-defined function:

$$f(n) = \begin{cases} f(1), & n = 1 \\ f(n-1) + d, & n > 1 \end{cases}$$

This is the **recursive definition** for an arithmetic sequence. Each term is defined by operations on the previous term.

Another way to write a recursive definition is to use subscript notation.

$$a_n = \begin{cases} a_1, & n = 1 \\ a_{n-1} + d, & n > 1 \end{cases}$$

With the notation, the subscript shows the number of the term.

C. Is the sequence 4, 7, 10, 13, 16, … arithmetic? If so, write the recursive definition for the sequence.

$a_1 = 4$ → 4, 7, 10, 13, 16

The common difference, d, is 3, so this is an arithmetic sequence.

The recursive definition for this sequence is

$$a_n = \begin{cases} 4, & n = 1 \\ a_{n-1} + 3, & n > 1. \end{cases}$$

Try It! **1.** Are the following sequences arithmetic? If so, what is the recursive definition, and what is the next term in the sequence?

a. 25, 20, 15, 10, …

b. 2, 4, 7, 12, 13, …

Activity Assess

EXAMPLE 2 Translate Between Recursive and Explicit Forms

A. Given the recursive definition $a_n = \begin{cases} 3, & n = 1 \\ a_{n-1} + 0.5, & n > 1 \end{cases}$ what is an explicit definition for the sequence?

An **explicit definition**, also written as $a_n = a_1 + d(n - 1)$, allows you to find any term in the sequence without knowing the previous term.

Use the recursive definition to find a pattern:

$a_1 = 3$

$a_2 = 3 + 0.5$

$a_3 = a_2 + 0.5 = [3 + 0.5] + 0.5 = 3 + 2(0.5)$

$a_4 = a_3 + 0.5 = [3 + 2(0.5)] + 0.5 = 3 + 3(0.5)$

So the explicit definition is $a_n = 3 + (n - 1)(0.5)$.

The first term has 0 common differences added. The second term has 1 common difference added to the first term. The third term has 2 common differences added, and so on.

USE STRUCTURE
Since $a_2 = a_1 + 0.5$, use substitution to simplify the expression.

In general, the explicit definition of an arithmetic sequence is $a_n = a_1 + d(n - 1)$.

B. Given the explicit definition $a_n = 16 - 3(n - 1)$, what is the recursive definition for the arithmetic sequence?

The common difference d is −3 and $a_1 = 16$.

The recursive definition is $a_n = \begin{cases} 16, & n = 1 \\ a_{n-1} - 3, & n > 1. \end{cases}$

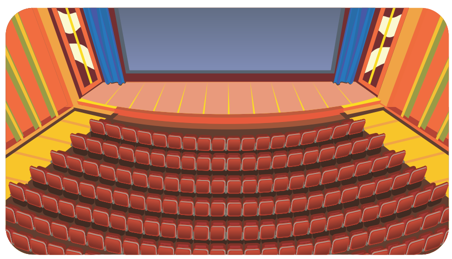
Try It! 2. a. For the recursive definition $a_n = \begin{cases} 45, & n = 1 \\ a_{n-1} - 2, & n > 1, \end{cases}$ what is the explicit definition?

b. For the explicit definition $a_n = 1 + 7(n - 1)$, what is the recursive definition?

APPLICATION

EXAMPLE 3 Solve Problems With Arithmetic Sequences

A high school auditorium has 18 seats in the first row and 26 seats in the fifth row. The number of seats in each row forms an arithmetic sequence.



A. What is the explicit definition for the sequence?

The problem states that $a_1 = 18$, $n = 5$, and $a_5 = 26$.

$a_n = a_1 + d(n - 1)$ .... Write the general explicit formula.

$26 = 18 + d(5 - 1)$ .... Substitute.

$26 = 18 + 4d$ .... Simplify.

$8 = 4d$ .... Simplify.

$2 = d$ .... Solve.

Each row has two more seats than the previous row.

The explicit definition is $a_n = 18 + 2(n - 1)$.

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EXAMPLE 3 CONTINUED

B. How many seats are in the twelfth row?

$a_n = 18 + 2(n - 1)$ Write the explicit formula.

$a_{12} = 18 + 2(12 - 1)$ Substitute 12 for n.

$a_{12} = 40$ Simplify.

The twelfth row has 40 seats.

Try It! **3.** Samantha is training for a race. The distances of her training runs form an arithmetic sequence. She runs 1 mi the first day and 2 mi the seventh day.

a. What is the explicit definition for this sequence?

b. How far does she run on day 19?

EXAMPLE 4 Find the Sum of an Arithmetic Series

A. What is the sum of the terms in the arithmetic sequence 1, 4, 7, 10, 13? What is a general formula for an arithmetic series?

A finite **series** is the sum of the terms in a finite sequence. A finite **arithmetic series** is the sum of the terms in an arithmetic sequence. For the sum of n numbers in a sequence, you can use a recursive formula, or simply add the terms.

$$S_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

$$\mathbf{1 + 4 + 7 + 10 + 13 = 35}$$

This represents a partial sum of a series because it is the sum of a finite number of terms, n, in the series.

To find the sum of a series with many terms, you can use an explicit definition.

STUDY TIP
The finite series represents an equation which can be added to a second related equation to help you.

Step 1 To find the explicit definition for the sum, use the Commutative Property of Addition and reverse the order of the terms in the recursive series.

$$S_5 = 13 + 10 + 7 + 4 + 1$$

Substitute the values from the series.

Step 2 Add the two expressions for the series, so you are adding the first term to the last term and the second term to the second-to-last term, and so on.

$$\begin{aligned} S_5 &= 1 + 4 + 7 + 10 + 13 \\ + S_5 &= 13 + 10 + 7 + 4 + 1 \\ \hline 2S_5 &= 14 + 14 + 14 + 14 + 14 \end{aligned}$$

Step 3 Simplify.

$$2 \cdot S_5 = 5(14)$$

$$S_5 = \frac{5(1 + 13)}{2}$$

Notice that 14 is the sum of the first and last terms, or $a_1 + a_5$.

Step 4 Write the general formula.

$$S_n = \frac{n(a_1 + a_n)}{2}$$

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EXAMPLE 4 CONTINUED

B. What is the sum of the following arithmetic sequence?
2, 6, 10, 14, 18, 22

$n = 6$ $\quad a_1 = 2$ $\quad a_n = 22$

Use the general formula to find the sum.

$$S_n = \frac{n(a_1 + a_n)}{2} \qquad S_n = \frac{6(2 + 22)}{2} = \frac{144}{2} = 72$$

The sum of the terms in the sequence is 72.

Try It! **4.** Find the sum of each arithmetic series.

a. series with 12 terms, $a_1 = 3$ and $a_{12} = 25$

b. $5 + 11 + 17 + 23 + 29 + 35 + 41$

CONCEPT Sigma Notation

The sum of n terms of a sequence can be written using **sigma notation**:

$$\sum_{i=1}^{n} a_i$$

The index i counts through the terms in the partial sum. Here it takes the values from 1 up to n, the last term in the partial series. The value of the ith term is a_i. You can use the explicit formula for the sequence in place of a_i.

EXAMPLE 5 Use Sigma Notation

A. What is $\sum_{i=1}^{9} 2i - 6$?

Find the sum by writing out all of the terms in the series.

$a_1 = 2(1) - 6 = -4$ $\quad a_4 = 2(4) - 6 = 2$ $\quad a_7 = 2(7) - 6 = 8$

$a_2 = 2(2) - 6 = -2$ $\quad a_5 = 2(5) - 6 = 4$ $\quad a_8 = 2(8) - 6 = 10$

$a_3 = 2(3) - 6 = 0$ $\quad a_6 = 2(6) - 6 = 6$ $\quad a_9 = 2(9) - 6 = 12$

$$\sum_{i=1}^{9} 2i - 6 = S_9 = -4 - 2 + 0 + 2 + 4 + 6 + 8 + 10 + 12 = 36$$

LOOK FOR RELATIONSHIPS
You can quickly see that the common difference in this series is 2.

B. How can you write the series $2 + 9 + 16 + \cdots + 79$ using sigma notation? What is the sum?

Step 1 Solve for n to find the number of terms in the series: $a_1 = 2$, $d = 7$, and $a_n = 79$.

$a_n = a_1 + d(n - 1)$ .... Write the explicit formula.

$79 = 2 + (7)(n - 1)$ .... Substitute a_n, a_1, and d.

$77 = (7)(n - 1)$ .... Simplify.

$11 = n - 1$ .... Simplify.

$n = 12$ .... Solve.

There are 12 terms in the series.

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EXAMPLE 5 CONTINUED

Step 2 Write the explicit formula for the series.

$$a_n = 2 + 7(n - 1) = 7n - 5$$

Step 3 Write using sigma notation.

The index i will count from 1 to $n = 12$.

The explicit definition gives $a_i = 7i - 5$.

$$\sum_{i=1}^{12} 7i - 5$$

Step 4 Find the sum of the series.

$$S_n = \frac{n(a_1 + a_n)}{2}$$

$$S_{12} = \frac{12(2 + 79)}{2} = 486$$

So the series can be written $\sum_{i=1}^{12} 7i - 5$ in sigma notation. The sum of the series is 486.

Try It! **5. a.** What is the sum of the series $\sum_{i=1}^{13} 3i + 2$?

b. How can you write the series 8 + 13 + 18 + . . . + 43 using sigma notation? What is the sum?

APPLICATION

EXAMPLE 6 Use a Finite Arithmetic Series

A pyramid of cans is on display in a supermarket. The top row has 1 can, the second row has 2 cans, and the third row has 3 cans. If there are 10 rows of cans, how many total cans were used to make the pyramid?

This is an arithmetic series where the common difference is 1.

Step 1 Find a_1 and a_{10}.

$$a_1 = 1$$

$$a_{10} = a_1 + d(n - 1)$$
$$= 1 + 1(10 - 1)$$
$$= 10$$

STUDY TIP
Since you know n, a_1, and a_{10}, you can use the explicit formula to find the sum of the series efficiently.

Step 2 Use the explicit formula for finding the sum of a series.

$$S_n = \frac{n(a_1 + a_n)}{2}$$

$$S_n = \frac{10(1 + 10)}{2} = 55$$

There are 55 cans in the display.

Try It! **6.** A flight of stairs gets wider as it descends. The top stair is 15 bricks across, the second stair is 17 bricks across, and the third stair is 19 bricks across. What is the total number of bricks used in all 16 stairs?

Concept Summary Assess

CONCEPT SUMMARY Arithmetic Sequences and Series

In an arithmetic sequence, each term is equal to the previous term plus a constant d, the common difference.

	Recursive Formula	Explicit Formula	Arithmetic Series
ALGEBRA	$a_n = \begin{cases} a_1, & n = 1 \\ a_{n-1} + d, & n > 1 \end{cases}$	$a_n = a_1 + d(n - 1)$	$S_n = \frac{n(a_1 + a_n)}{2}$
NUMBERS	For $a_1 = 1$ and $d = 7$, $a_2 = 1 + 7 = 8$ $a_3 = 8 + 7 = 15$ $a_4 = 15 + 7 = 22$ ....and so on	For $a_1 = 90$ and $d = -4$, $a_2 = 90 + 1(-4) = 86$ $a_3 = 90 + 2(-4) = 82$ $a_4 = 90 + 3(-4) = 78$ ....and so on	$\sum_{i=1}^{8} 5i - 2$ $3 + 8 + 13 + 18 + 23 + 28 + 33 + 38 = 164$ or $S_8 = \frac{8(3 + 38)}{2} = 164$

Do You UNDERSTAND?

1. ESSENTIAL QUESTION What is an arithmetic sequence, and how do you represent and find its terms and their sums?

2. **Vocabulary** How do arithmetic sequences differ from arithmetic series?

3. **Error Analysis** A student claims the sequence 0,1,3, 6, … is an arithmetic sequence, and the next number is 10. What error did the student make?

4. **Communicate Precisely** How would you tell someone how to calculate $\sum_{n=1}^{5} (2n + 1)$?

Do You KNOW HOW?

Find the common difference and the next three terms of each arithmetic sequence.

5. $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \ldots$

6. 6, 1, −4, −9, −14, . . .

7. 215, 227, 239, 251, . . .

8. −4, −5, −6, −7, . . .

9. 4.1, 6.3, 8.5, 10.7, . . .

10. −17, −9, −1, 7, 15, . . .

11. In June, you start a holiday savings account with a deposit of \$30. You increase each monthly deposit by \$4 until the end of the year. How much money will you have saved by the end of December?

PRACTICE & PROBLEM SOLVING

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UNDERSTAND

12. **Use Structure** Write an arithmetic sequence with at least four terms, and describe it using both an explicit and recursive definition.

13. **Error Analysis** Alex says the common difference for an arithmetic sequence is always negative because of the definition of *difference*. Why is he wrong? Write an arithmetic sequence to show he is wrong.

14. **Use Structure** A company will pay Becky \$120 for her first sale. For each sale after that, they will pay an extra \$31.50 per sale. So, she will make \$151.50 for the second sale, \$183 for the third sale, and so on. How many sales will Becky have to make to earn at least \$2,000?

15. **Higher Order Thinking** Felipe and Gregory are given the arithmetic sequence −1, 6, 13, ... Gregory wrote the explicit definition $a_n = -1 + 7(n - 1)$ for the sequence. Felipe wrote the definition as $a_n = 7n - 8$. Which one of them is correct? Explain.

16. **Model With Mathematics** Suppose you are building 10 steps with 8 concrete blocks in the top step and 80 blocks in the bottom step. If the number of blocks in each step forms an arithmetic sequence, find the total number of concrete blocks needed to build the steps.

17. **Model With Mathematics** With her half-marathon quickly approaching, Talisa decides to train every day up to the day of the race. She plans to run 2 mi the first day and 3.2 mi the fifth day.

    a. What is the explicit definition for this sequence?

    b. Which day of training will she run the distance of a half-marathon (13.1 mi)?

PRACTICE

Are the following sequences arithmetic? If so, what is the common difference? What is the next term in the sequence? SEE EXAMPLE 1

18. 10, 20, 30, 40, . . .

19. 97, 86, 75, 64, . . .

20. 1, 4, 9, 16, . . .

21. 3, 7, 11, 15, . . .

Translate between the recursive and explicit definitions for each sequence. SEE EXAMPLE 2

22. $a_n = \begin{cases} 2, & n = 1 \\ a_{n-1} + 2, & n > 1 \end{cases}$

23. $a_n = -2 + 7(n - 1)$

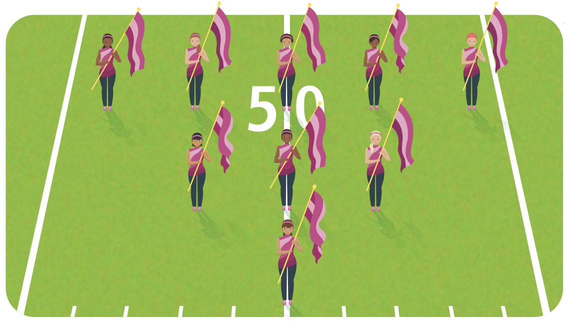
24. $a_n = \frac{1}{8}(n - 1)$

25. $a_n = \begin{cases} -4, & n = 1 \\ a_{n-1} - 4, & n > 1 \end{cases}$

26. The members of a school's color guard begin their performance in a pyramid formation. The first row has 1 member, and the third row has 5 members. SEE EXAMPLE 3

    a. What is the explicit definition for this sequence?

    b. How many members are in the eighth row?



Find the sum of an arithmetic series with the given number of terms, a_1, and a_n. SEE EXAMPLE 4

27. 10 terms, $a_1 = 4$, $a_{10} = 31$

28. 15 terms, $a_1 = 17$, $a_{15} = 129$

What is the sum of each of the following series? SEE EXAMPLE 5

29. $\sum_{n=1}^{11} (3 + 2n)$

30. $\sum_{n=1}^{12} \left(\frac{n}{2} - 9\right)$

31. The number of seats in each row of an auditorium increases as you go back from the stage. The front row has 24 seats, the second row has 29 seats, and the third row has 34 seats. If there are 35 rows, how many seats are in the auditorium? SEE EXAMPLE 6

PRACTICE & PROBLEM SOLVING

Practice Tutorial

Mixed Review Available Online

APPLY

32. **Make Sense and Persevere** A piece of tile artwork is in the shape of a triangle. The top row has 1 tile, the second row has 2 tiles, and the third row has 3 tiles. If there are 14 rows of tiles, how many tiles were used to make the artwork?

33. **Model With Mathematics** A race car driver travels 34 ft in the first second of a race. If the driver travels 3.5 additional feet each subsequent second, how many feet did the driver travel in 52 s?

34. **Construct Arguments** A school board committee has decided to spend its annual technology budget this year on 90 student laptops and plans to buy 40 new laptops each year from now on.

a. The school board decided that each student in the school should have access to a laptop in the next ten years. If there are 500 students, will the technology coordinator meet this goal? Explain.

b. What are some pros and cons of buying student laptops in this manner? If you could change the plan, would you? If so, how would you change it?

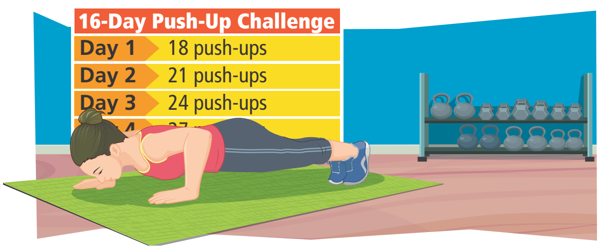
35. **Make Sense and Persevere** On October 1, Nadia starts a push-up challenge by doing 18 push-ups. On October 2, she does 21 push-ups. On October 3, she does 24 push-ups. She continues until October 16, when she does the final push-ups in the challenge.

a. Write an explicit definition to model the number of push-ups Nadia does each day.

b. Write a recursive definition to model the number of push-ups Nadia does each day.

c. How many push-ups will Nadia do on October 16?

d. What is the total number of push-ups Nadia does from October 1 to October 16?



ASSESSMENT PRACTICE

36. Prior to 1994, the Winter Olympic Games were held in the same year as the Summer Olympic Games. The 1994 Winter Olympics were held in Lillehammer, Norway, and have taken place every four years since.

A recursive definition for a sequence that models the year of the Winter Olympics is given.

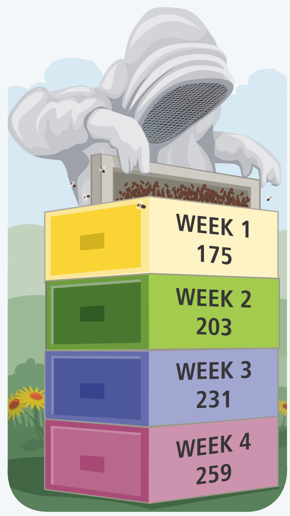
$$a_n = \begin{cases} 1994, & n = 1 \\ a_{n-1} + 4, & n > 1 \end{cases}$$

Write an explicit formula for the same sequence. F-BF.1.2

37. **SAT/ACT** Tamika is selling magazines door to door. On her first day, she sells 12 magazines, and she intends to sell 5 more magazines per day than on the previous day. If she meets her goal and sells magazines for a total of 10 days, how many magazines would she sell?

Ⓐ 314 Ⓑ 345 Ⓒ 415 Ⓓ 474 Ⓔ 505

38. **Performance Task** The chart shows the population of Edgar's beehive over the first four weeks. Assume the population will continue to grow at the same rate.



Part A Write an explicit definition for the sequence.

Part B If Edgar's bees have a mass of 1.5 g each, what will the total mass of all his bees be in 12 wk?

Part C When the colony reaches 1,015 bees, Edgar's beehive will not be big enough for all of them. In how many weeks will the bee population be too large?