3-4 Dividing Polynomials

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I CAN… divide polynomials.

VOCABULARY

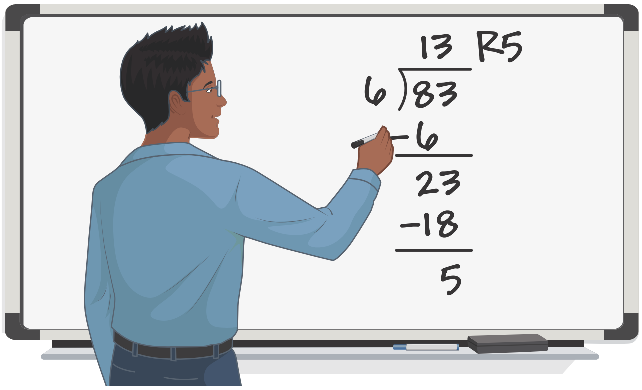
- Factor Theorem
- Remainder Theorem
- synthetic division

MAFS.912.A-APR.4.6–Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
Also A-SSE.1.2, A-APR.2.2
MAFS.K12.MP.2.1, MP.6.1, MP.7.1

Activity   Assess

EXPLORE & REASON

Benson recalls how to divide whole numbers by solving a problem with 6 as the divisor and 83 as the dividend. He determines that the quotient is 13 with remainder 5.



A. Explain the process of long division using Benson's example.

B. How can you express the remainder as a fraction?

C. **Use Structure** Use the results of the division problem to write two expressions for 83 that include the divisor, quotient, and remainder.

ESSENTIAL QUESTION

How can you divide polynomials?

EXAMPLE 1 Use Long Division to Divide Polynomials

How can you use long division to divide $P(x)$ by $D(x)$? Write the polynomial $P(x)$ in terms of the quotient and remainder.

A. **Let $P(x) = x^3 + 5x^2 + 6x + 9$ and $D(x) = x + 3$.**

Long division of polynomials is similar to long division of numbers.

$$x^2 + 2x$$
$$x + 3\,\overline{)\,x^3 + 5x^2 + 6x + 9}$$

Divide the leading terms: $x^3 \div x = x^2$.

$$-\,(x^3 + 3x^2)$$

Multiply: $x^2(x + 3) = x^3 + 3x$. Then subtract.

$$2x^2 + 6x + 9$$

Divide the leading terms again: $2x^2 \div x = 2x$.

$$-(2x^2 + 6x)$$

Multiply: $2x(x + 3) = 2x + 6x$.

$$9$$

Subtract. The remainder is 9.

When you divide polynomials, you can express the relationship of the quotient and remainder to the dividend and divisor in two ways.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

$$\frac{x^3 + 5x^2 + 6x + 9}{x + 3} = x^2 + 2x + \frac{9}{x + 3}$$

$$\text{dividend} = \text{quotient} \times \text{divisor} + \text{remainder}$$

$$x^3 + 5x^2 + 6x + 9 = (x^2 + 2x)(x + 3) + 9$$

LOOK FOR RELATIONSHIPS
Compare the long division of these two polynomials to this numerical long division problem.

$$\begin{array}{r} 120 \\ 13\overline{)1{,}569} \\ \underline{-13} \\ 26 \\ \underline{-26} \\ 09 \end{array}$$

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Activity Assess

EXAMPLE 1 CONTINUED

B. Let $P(x) = 8x^3 + 27$ and $D(x) = 2x + 3$.

The dividend is a cubic polynomial with no first- or second-degree term.

$$\begin{array}{r}
4x^2 - 6x + 9 \\
2x + 3\,\overline{)\,8x^3 + 0x^2 + 0x + 27} \\
\underline{-(8x^3 + 12x^2)} \\
-12x^2 + 0x + 27 \\
\underline{-(-12x^2 - 18x)} \\
18x + 27 \\
\underline{-(18x + 27)} \\
0
\end{array}$$

Use 0 as the coefficient for the missing 1st- and 2nd-degree terms.

The remainder is 0. This means the divisor is a factor of the dividend.

GENERALIZE
When the remainder is 0, you can use the results of the long division to write $P(x)$ in factored form.

So, $\frac{8x^3 + 27}{2x + 3} = 4x^2 - 6x + 9$ and $8x^3 + 27 = (2x + 3)(4x^2 - 6x + 9)$.

Try It! **1.** Use long division to divide the polynomials. Then write the dividend in terms of the quotient and remainder.

a. $x^3 - 6x^2 + 11x - 6$ divided by $x^2 - 4x + 3$

b. $16x^4 - 85$ divided by $4x^2 + 9$

EXAMPLE 2 Use Synthetic Division to Divide by $x - a$

What is $2x^3 - 7x^2 - 4$ divided by $x - 3$? Use synthetic division.

Synthetic division is a method used to divide a polynomial by a linear expression in the form $x - a$. Note that the leading coefficient of the divisor is 1, and that a is the zero of the divisor.

Step 1 To change from long division format to synthetic division format, write only the zero of the divisor and the coefficients of the dividend.

$$\begin{array}{r|rrrr}
x - 3 & 2x^3 & -\,7x^2 & +\,0x & -\,4 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
3 & 2 & -7 & 0 & -4
\end{array}$$

zero of the divisor, a

coefficients of the dividend

Separate a from the dividend.

Step 2 Bring down the first coefficient. Multiply the zero of the divisor by the first coefficient. Add the result to the second coefficient.

$$\begin{array}{r|rrrr}
3 & 2 & -7 & 0 & -4 \\
 & & 6 & & \\
\hline
 & 2 & -1 & &
\end{array}$$

$3 \cdot 2 = 6$

$-7 + 6 = -1$

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Activity | Assess

EXAMPLE 2 CONTINUED

Step 3 Repeat the process until all the columns are complete.

$$\begin{array}{r|rrrr} 3 & 2 & -7 & 0 & -4 \\ & & 6 & -3 & \\ \hline & 2 & -1 & -3 & \end{array}$$

$3 \cdot -1 = -3$

$$\begin{array}{r|rrrr} 3 & 2 & -7 & 0 & -4 \\ & & 6 & -3 & -9 \\ \hline & 2 & -1 & -3 & -13 \end{array}$$

$3 \cdot -3 = -9$

Step 4 Use the numbers in the last row to write the quotient and remainder.

$$\begin{array}{r|rrrr} 3 & 2 & -7 & 0 & -4 \\ & & 6 & -3 & -9 \\ \hline & 2 & -1 & -3 & -13 \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ & 2x^2 & -\ 1x & -\ 3 & -\ \frac{13}{x-3} \end{array}$$

coefficients of the quotient

remainder

Since the dividend is cubic and the divisor is linear, the quotient is quadratic.

So, $\frac{2x^3 - 7x^2 - 4}{(x - 3)} = 2x^2 - x - 3 - \frac{13}{x - 3}$.

COMMON ERROR
Remember to keep track of all positive and negative signs when multiplying.

USE STRUCTURE
You can use the result to write the dividend $2x^3 - 7x - 4$ in the form $(2x^2 - x - 3)(x - 3) - 13$.

Try It! **2.** Use synthetic division to divide $3x^3 - 5x + 10$ by $x - 1$.

CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Relate $P(a)$ to the Remainder of $P(x) \div (x - a)$

How is the value of $P(a)$ related to the remainder of $P(x) \div (x - a)$?

To explore this question, let $P(x) = x^3 + 10x^2 + 29x + 24$. Use synthetic division to divide $P(x)$ by $x + 5$.

To identify the value of a, write the divisor $x + 5$ in the form $x - a$.

$x + 5 = x - (-5)$ — $a = -5$

$$\begin{array}{r|rrrr} -5 & 1 & 10 & 29 & 24 \\ & & -5 & -25 & -20 \\ \hline & 1 & 5 & 4 & 4 \end{array}$$

The remainder is 4.

The quotient is $x^2 + 5x + 4$.

So, $P(x) = (x^2 + 5x + 4)(x + 5) + 4$. Use this form to evaluate $P(-5)$.

$$\begin{aligned} P(-5) &= [((-5)^2 + 5(-5) + 4)(-5 + 5)] + 4 \\ &= [(25 - 25 + 4)(0)] + 4 \\ &= 0 + 4 \\ &= 4 \end{aligned}$$

-5 is the zero of the divisor, so the product of the quotient and the divisor is 0.

So, $P(-5)$ is the remainder, 4, of $P(x)$ divided by $x - (-5)$.

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Activity Assess

EXAMPLE 3 CONTINUED

In general, dividing $P(x)$ by $x - a$ results in a quotient $Q(x)$ and a remainder r.

$$P(x) = Q(x)(x - a) + r$$
$$P(a) = Q(x)(a - a) + r$$
$$= Q(x)(0) + r$$
$$= r$$

Evaluating $P(x)$ at a, the zero of the divisor, shows that $P(a) = r$.

So, the for a polynomial $P(x)$ the value of $P(a)$ is equal to the remainder of the division $P(x) \div (x - a)$.

Try It! **3.** Use synthetic division to show that the remainder of $f(x) = x^3 + 8x^2 + 12x + 5$ divided by $x + 2$ is equal to $f(-2)$.

CONCEPT Remainder Theorem and Factor Theorem

The **Remainder Theorem** states that if a polynomial $P(x)$ is divided by $x - a$, the remainder is $P(a)$.

When $x - a$ is a factor of $P(x)$, we can show that $P(a) = 0$.

$$P(x) = Q(x)(x - a)$$
$$P(a) = Q(x)(a - a)$$
$$P(a) = 0$$

Conversely, when $P(a) = 0$, we can show that $x - a$ is a factor.

$$P(x) = Q(x)(x - a) + r$$
$$P(x) = Q(x)(x - a) + P(a)$$
$$P(x) = Q(x)(x - a) + 0$$
$$P(x) = Q(x)(x - a)$$

Use the Remainder Theorem.

The Factor Theorem formalizes these two points.

The **Factor Theorem** states that the expression $x - a$ is a factor of a polynomial $P(x)$ if and only if $P(a) = 0$.

APPLICATION

EXAMPLE 4 Use the Remainder Theorem to Evaluate Polynomials

The population of tortoises on an island is modeled by the function $P(x) = -x^3 + 6x^2 + 12x + 325$ where x is the number of years since 2015. Use the Remainder Theorem to estimate the population in 2023.

Use synthetic division to find $P(a)$ when $a = 8$.

8	−1	6	12	325
		−8	−16	−32
	−1	−2	−4	293

The estimated population in 2023 is 293 tortoises.

Try It! **4.** A technology company uses the function $R(x) = -x^3 + 12x^2 + 6x + 80$ to model expected annual revenue, in thousands of dollars, for a new product, where x is the number of years after the product is released. Use the Remainder Theorem to estimate the revenue in year 5.

EXAMPLE 5 Check Whether $x - a$ is a Factor of $P(x)$

How can you use the Remainder and Factor Theorems to determine whether the given binomial is a factor of $P(x)$? If it is a factor, write the polynomial in factored form.

A. $P(x) = x^4 - 8x^3 + 16x^2 - 23x - 6$**; binomial:** $x - 6$

The binomial $x - 6$ is a factor of $P(x)$ if 6 is a zero of $P(x)$.

Method 1 Use synthetic substitution.

6	1	−8	16	−23	−6
		6	−12	24	6
	1	−2	4	1	0

Method 2 Use direct substitution.

$$P(6) = 6^4 - 8(6^3) + 16(6^2) - 23(6) - 6$$
$$= 1{,}296 - 1{,}728 + 576 - 138 - 6$$
$$= 0$$

Because $P(6) = 0$, you can use the Factor Theorem to conclude that $x - 6$ is a factor of $P(x)$: $P(x) = (x^3 - 2x^2 + 4x + 1)(x - 6)$.

B. $P(x) = x^5 - 5x^3 + 9x^2 - x + 3$**; binomial:** $x + 3$

Method 1 Use synthetic substitution.

−3	1	0	−5	9	−1	3
		−3	9	−12	9	−24
	1	−3	4	−3	8	−21

COMMON ERROR
When using synthetic division, remember to include 0 coefficients for any missing terms.

Method 2 Use direct substitution.

$$P(-3) = (-3)^5 - 5((-3)^3) + 9((-3)^2) - (-3) + 3$$
$$P(-3) = -243 + 135 + 81 + 3 + 3$$
$$P(-3) = -21$$

Because −3 is a not a zero of $P(x)$, you can use the Factor Theorem to conclude that $x + 3$ is not a factor of $P(x)$.

Try It! **5.** Use the Remainder and Factor Theorems to determine whether the given binomial is a factor of $P(x)$.

a. $P(x) = x^3 - 10x^2 + 28x - 16$; binomial: $x - 4$

b. $P(x) = 2x^4 + 9x^3 - 2x^2 + 6x - 40$; binomial: $x + 5$

Concept Summary Assess

CONCEPT SUMMARY Dividing Polynomials

Example: Divide $x^3 - 8x^2 - 5x - 30$ by $x - 9$.

LONG DIVISION

Can be used for any polynomial division.

$$\begin{array}{r} x^2 + x + 4 \\ x - 9\overline{)x^3 - 8x^2 - 5x - 30} \\ \underline{-(x^3 - 9x^2)} \\ x^2 - 5x - 30 \\ \underline{-(x^2 - 9x)} \\ 4x - 30 \\ \underline{-(4x - 36)} \\ 6 \end{array}$$

SYNTHETIC DIVISION

Most readily used when the divisor is linear and its leading coefficient is 1.

$$\begin{array}{r|rrrr} 9 & 1 & -8 & -5 & -30 \\ & & 9 & 9 & 36 \\ \hline & 1 & 1 & 4 & 6 \end{array}$$

Either method shows that $x^3 - 8x^2 - 5x - 30 = (x^2 + x + 4)(x - 9) + 6$

REMAINDER THEOREM

If a polynomial $P(x)$ is divided by a linear divisor $x - a$, the remainder is $P(a)$.

$P(x) = x^3 - 2x + 1$ divided by $x - 2$ has remainder 5.

$$\begin{array}{r|rrrr} 2 & 1 & 0 & -2 & 1 \\ & & 2 & 4 & 4 \\ \hline & 1 & 2 & 2 & 5 \end{array}$$

$P(2) = 5$

FACTOR THEOREM

The binomial $x - a$ is a factor of $P(x)$ if and only if $P(a) = 0$.

$P(x) = 2x^4 - 5x^3 - 12x^2 + x - 4$ divided by $x - 4$ has remainder 0.

$$\begin{array}{r|rrrrr} 4 & 2 & -5 & -12 & 1 & -4 \\ & & 8 & 12 & 0 & 4 \\ \hline & 2 & 3 & 0 & 1 & 0 \end{array}$$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you divide polynomials?

2. **Error Analysis** Ella said the remainder of $x^3 + 2x^2 - 4x + 6$ divided by $x + 5$ is 149. Is Ella correct? Explain.

3. **Look for Relationships** You divide a polynomial $P(x)$ by a linear expression $D(x)$. You find a quotient $Q(x)$ and a remainder $R(x)$. How can you check your work?

Do You KNOW HOW?

4. Use long division to divide $x^4 - 4x^3 + 12x^2 - 3x + 6$ by $x^2 + 8$.

5. Use synthetic division to divide $x^3 - 8x^2 + 9x - 5$ by $x - 3$.

6. Use the Remainder Theorem to find the remainder of $2x^4 + x^2 - 10x - 1$ divided by $x + 2$.

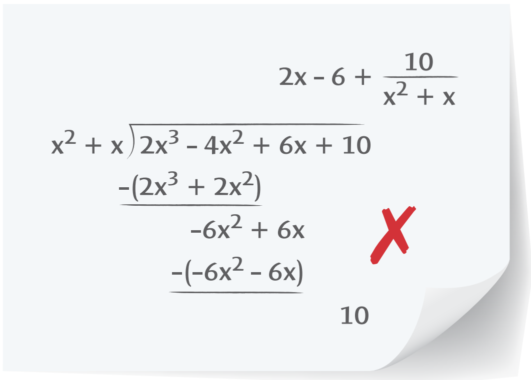
7. Is $x + 9$ a factor of the polynomial $P(x) = x^3 + 11x^2 + 15x - 27$? If so, write the polynomial as a product of two factors. If not, explain how you know.

UNDERSTAND

8. **Reason** Write a polynomial division problem with a quotient of $x^2 - 5x + 7$ and a remainder of 2. Explain your reasoning. How can you verify your answer?

9. **Communicate Precisely** Show that $x - 3$ and $x + 5$ are factors of $x^4 + 2x^3 - 16x^2 - 2x + 15$. Explain your reasoning.

10. **Error Analysis** Alicia divided the polynomial $2x^3 - 4x^2 + 6x + 10$ by $x^2 + x$. Describe and correct the error Alicia made in dividing the polynomials.



11. **Higher Order Thinking** When dividing polynomial $P(x)$ by polynomial $d(x)$, the remainder is $R(x)$. The remainder can also be written as $\frac{R(x)}{d(x)}$. How can you use the degrees of $R(x)$ and $d(x)$ to determine whether you are finished dividing?

12. **Look for Relationships** When dividing polynomial $P(x)$ by polynomial $x - n$, the remainder is 0. When graphing $P(x)$, what is an x-intercept of the graph?

13. **Reason** When dividing $x^3 + nx^2 + 4nx - 6$ by $x + 3$, the remainder is –48. What is the value of n?

14. **Mathematical Connections** Use polynomial long division to divide $8x^3 + 27$ by $2x + 3$. How can you use multiplication to check your answer? Show your work.

PRACTICE

Use long division to divide. SEE EXAMPLE 1

15. $x^3 + 5x^2 - x - 5$ divided by $x - 1$

16. $2x^3 + 9x^2 + 10x + 3$ divided by $2x + 1$

17. $3x^3 - 2x^2 + 7x + 9$ divided by $x^2 - 3x$

18. $2x^4 - 6x^2 + 3$ divided by $2x - 6$

Use synthetic division to divide. SEE EXAMPLE 2

19. $x^4 - 25x^2 + 144$ divided by $x - 4$

20. $x^3 + 6x^2 + 3x - 10$ divided by $x + 5$

21. $x^5 + 2x^4 - 3x^3 + x - 1$ divided by $x + 2$

22. $-x^4 + 7x^3 + x^2 - 2x - 12$ divided by $x - 3$

23. Use synthetic division to show that the remainder of $f(x) = x^4 - 6x^3 - 33x^2 + 46x + 75$ divided by $x - 9$ is $P(9)$. SEE EXAMPLE 3

Use the Remainder Theorem to evaluate each polynomial for the given value of x. SEE EXAMPLE 4

24. $f(x) = x^3 + 9x^2 + 3x - 7;\ x = -5$

25. $f(x) = 2x^3 - 3x^2 + 4x + 13;\ x = 3$

26. $f(x) = -x^4 + 2x^3 - x^2 + 4x + 8;\ x = -2$

27. $f(x) = x^5 - 3x^4 - 2x^3 + x^2 - 2x - 1;\ x = 4$

Is each given binomial a factor of the given polynomial? If so, write the polynomial as a product of two factors. SEE EXAMPLE 5

28. polynomial: $P(x) = 8x^3 - 10x^2 + 28x - 16$; binomial: $x - 3$

29. polynomial: $P(x) = 4x^4 - 9x^3 - 7x^2 - 2x + 25$; binomial: $x + 4$

30. polynomial: $P(x) = -x^5 + 12x^3 + 6x^2 - 23x + 1$; binomial: $x - 2$

31. polynomial: $P(x) = 2x^3 + 3x^2 - 8x - 12$; binomial: $2x + 3$

PRACTICE & PROBLEM SOLVING

APPLY

32. Model With Mathematics Darren is placing shipping boxes in a storage unit with a floor area of $x^4 + 5x^3 + x^2 - 20x - 14$ square units. Each box has a volume of $x^3 + 10x^2 + 29x + 20$ cubic units and can hold a stack of items with a height of $x + 5$ units.

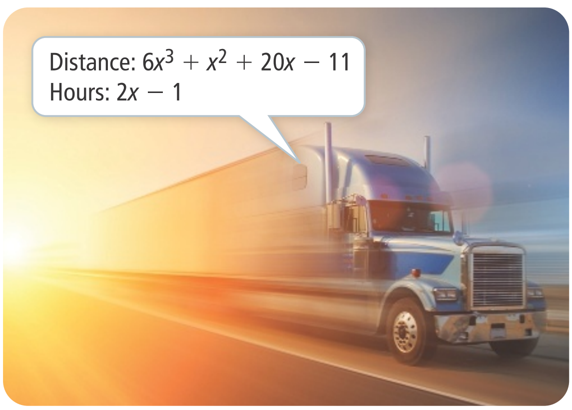
a. How much floor space will each box cover?

b. What is the maximum number of boxes Darren can place on the floor of the storage unit?

c. Assume Darren places the maximum number of boxes on the floor of the storage unit, with no overlap. How much of the floor space is not covered by a box?

33. Reason Lauren wants to determine the length and height of her DVD stand. The function $f(x) = x^3 + 14x^2 + 57x + 72$ represents the volume of the DVD stand, where the width is $x + 3$ units. What are possible dimensions for the length and height of the DVD stand? Explain.

34. Make Sense and Persevere A truck traveled $6x^3 + x^2 + 20x - 11$ miles in $2x - 1$ hours. At what rate did the semi-truck travel? (*Hint:* Use the formula $d = rt$, where d is the distance, r is the rate, and t is the time.)



ASSESSMENT PRACTICE

35. When polynomial $P(x)$ is divided by the linear factor $x - n$, the remainder is 0. What can you conclude? Select all that apply. A-APR.2.2

☐ **A.** $P(x) = 0$

☐ **B.** $P(n) = 0$

☐ **C.** $P(-n) = 0$

☐ **D.** $x - n$ is a factor of $P(x)$.

☐ **E.** $x + n$ is a factor of $P(x)$.

36. SAT/ACT $x + 3$ is a factor of the polynomial $x^3 + 2x^2 - 5x + n$. What is the value of n?

Ⓐ −6

Ⓑ −3

Ⓒ −2

Ⓓ 3

Ⓔ 6

37. Performance Task The table shows some quotients of the polynomial $x^n - 1$ divided by the linear factor $x - 1$.

Dividend	Divisor	Quotient
$x^2 - 1$	$x - 1$	$x + 1$
$x^3 - 1$	$x - 1$	$x^2 + x + 1$
$x^4 - 1$	$x - 1$	
$x^5 - 1$	$x - 1$	
$x^6 - 1$	$x - 1$	

Part A Use long division or synthetic division to find the missing quotients to complete the table.

Part B Look for a pattern. Then describe the pattern when $x^n - 1$ is divided by $x - 1$.

Part C Use the pattern to find the quotient when $x^{10} - 1$ is divided by $x - 1$.