6-1 Key Features of Exponential Functions

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Activity Assess

I CAN… recognize the key features of exponential functions.

VOCABULARY

- decay factor
- exponential decay function
- exponential function
- exponential growth function
- growth factor

MAFS.912.F-IF.2.4–For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity. Also F-IF.2.5, F-IF.3.9, F-BF.2.3, F-LE.2.5

MAFS.K12.MP.2.1, MP.4.1, MP.7.1

EXPLORE & REASON

Margaret investigates three functions: $y = 3x$, $y = x^3$, and $y = 3^x$. She is interested in the differences and ratios between consecutive y-values. Here is the table she started for $y = 3x$.

Investigating y = 3x			
x	y	Difference between y-values	Ratio between y-values
1	3		
2	6	$6 - 3 = 3$	$\frac{6}{3} = 2$
3	9	$9 - 6 = 3$	$\frac{9}{6} = 1.5$
4	12	$12 - 9 = 3$	$\frac{12}{9} \approx 1.33$

A. Create tables like Margaret's for all three functions and fill in more rows.

B. Which functions have a constant difference between consecutive y-values? Constant ratio?

C. **Use Structure** Which of these three functions will have y-values that increase the fastest as x increases? Why?

ESSENTIAL QUESTION

How do graphs and equations reveal key features of exponential growth and decay functions?

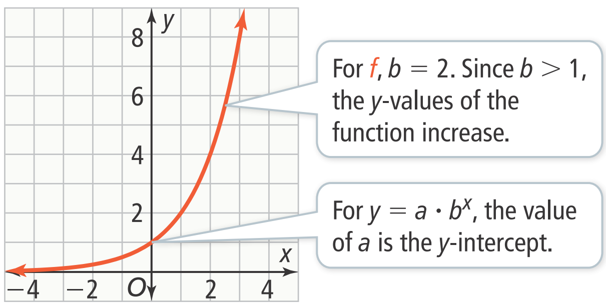
EXAMPLE 1 Identify Key Features of Exponential Functions

What are the key features of each function? Include domain, range, intercepts, asymptotes, and end behavior.

An **exponential function** is any function of the form $y = a \cdot b^x$ where a and b are constants with $a \neq 0$, and $b > 0$, $b \neq 1$.

A. $f(x) = 2^x$

Graphing $y = a \cdot b^x$	
x	$f(x) = 2^x$
−2	0.25
−1	0.5
0	1
1	2
2	4



Domain: all real numbers
Range: $\{y \mid y > 0\}$
y-intercept: 1;

Asymptote: x-axis
End Behavior:
As $x \to -\infty$, $y \to 0$. As $x \to \infty$, $y \to \infty$.

STUDY TIP

Recall your investigation of the ratios of consecutive y-values in the Explore & Reason activity. That ratio, which was 3 for the function $y = 3^x$, is equal to the value of b in the equation $y = a \cdot b^x$.

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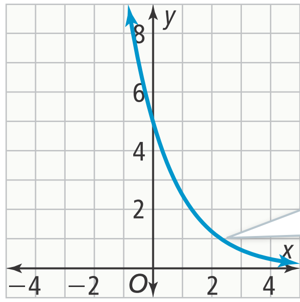
EXAMPLE 1 CONTINUED

B. $g(x) = 5\left(\frac{1}{2}\right)^x$

Graphing $y = a \cdot b^x$

x	$g(x) = 5\left(\frac{1}{2}\right)^x$
−2	20
−1	10
0	5
1	2.5
2	1.25



For g, $b = \frac{1}{2}$. Since $b < 1$, the y-values of the function decrease.

Domain: all real numbers
Range: $\{y \mid y > 0\}$
y-intercept: 5;

Asymptote: x-axis
End Behavior: As $x \to -\infty$, $y \to \infty$.
As $x \to \infty$, $y \to 0$.

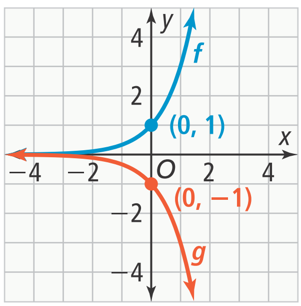
Try It! **1.** Graph $f(x) = 4(0.5)^x$. What are the domain, range, intercepts, asymptote, and the end behavior for this function?

EXAMPLE 2 Graph Transformations of Exponential Functions

Graph each function. Describe the graph in terms of transformations of the parent function $f(x) = 3^x$. How do the asymptote and intercept of the given function compare to the asymptote and intercept of the parent function?

A. $g(x) = -3^x$

When the sign of a changes, the function is reflected across the x-axis.



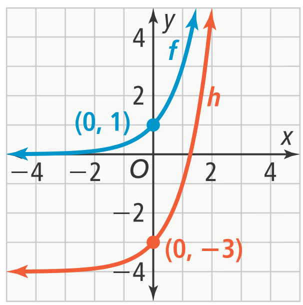
$g(x) = -3^x = -f(x)$

The intercept changes from a to $-a$.

The asymptote of the function does not change. It is still the x-axis.

B. $h(x) = 3^x - 4$

When adding a constant k, the function shifts vertically by k units.



$h(x) = 3^x - 4 = f(x) - 4$

The intercept changes from 1 to $1 + K$.

The asymptote of the function also changes. It is $y = k$ or $y = -4$.

COMMON ERROR
You may confuse reflection across the axes. Recall that if the y-value is multiplied by −1 (as in this case, with $g(x) = -3^x$), the reflection is across the x-axis. Each y-value is replaced by its opposite.

Try It! **2.** How do the asymptote and intercept of the given function compare to the asymptote and intercept of the function $f(x) = 5^x$?

a. $g(x) = 5^{x+3}$ **b.** $h(x) = 5^{-x}$

CONCEPTUAL UNDERSTANDING

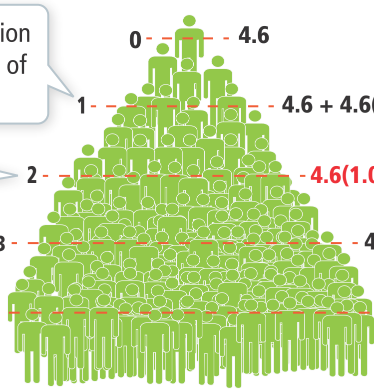
EXAMPLE 3 Model with Exponential Functions

The population of a large city was about 4.6 million in the year 2010 and grew at a rate of 1.3% for the next four years.

A. What exponential function models the population of the city over that 4-year period?

Compute the population for the first few years to look for a pattern.

t (years since 2010) **p (population in millions)**



The population can be modeled by the exponential function:

population after t years | population in 2010 | growth factor | years since 2010

$$P = 4.6(1.013)^t$$

B. If the population continues to grow at the same rate, what will the population be in 2040?

To find the population in 2040, solve the equation for $t = 30$:

$$P = 4.6(1.013)^{30} \approx 6.78.$$

In 2040, the population will be about 6.78 million.

USE STRUCTURE
Exponential functions of the form $y = a \cdot b^x$ involve repeated multiplication by the factor b. To understand how to model population with an exponential function, look for repeated multiplication in your computations.

Try It! **3.** A factory purchased a 3D printer in 2010. The value of the printer is modeled by the function $f(x) = 30(0.93)^x$, where x is the number of years since 2010.

a. What is the value of the printer after 10 years?

b. Does the printer lose more of its value in the first 10 years or in the second 10 years after it was purchased?

CONCEPT Exponential Growth and Decay Models

Exponential growth and **exponential decay** functions model quantities that increase or decrease by a fixed percent during each time period. Given an initial amount a and the rate of increase or decrease r, the amount $A(t)$ after t time periods is given by:

Exponential Growth Model

$$A(t) = a(1 + r)^t$$

$$a > 0,\ b > 1,\ b = 1 + r$$

Exponential Decay Model

$$A(t) = a(1 - r)^t$$

$$a > 0,\ 0 < b < 1,\ b = 1 - r$$

The **growth or decay factor** is equal to b, and is the ratio between two consecutive y-values.

EXAMPLE 4 Interpret an Exponential Function

A car was purchased for \$24,000. The function $y = 24 \cdot 0.8^x$ can be used to model the value of the car (in thousands of dollars) x years after it was purchased.

A. Does the function represent exponential growth or decay?

$$y = 24 \cdot 0.8^x$$

$b = 0.8$, so $b < 1$ and the function represents exponential decay.

B. What is the rate of decay for this function? What does it mean?

$$b = 1 - r$$

$$0.8 = 1 - r$$

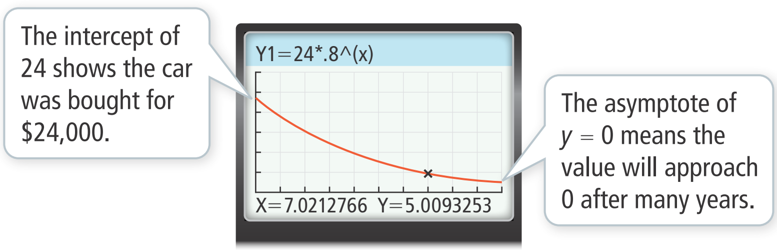
$$r = 0.2$$

The rate of decay is 0.2, or 20%. This means that the value of the car decreases by 20% each year.

STUDY TIP

For a function of the form $y = a \cdot b^x$, if $b > 1$, the function is increasing. If $0 < b < 1$, the function is decreasing.

C. Graph the function on a reasonable domain. What do the y-intercept and asymptote represent? When will the value of the car be about \$5,000?



Find the value of x when $y = 5$.

The graph (approximately) passes through the point (7, 5). This means that the value of the car will be about \$5,000 after 7 years.

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Try It! **4.** Two-hundred twenty hawks were released into a region on January 2, 2016. The function $f(x) = 220(1.05)^x$ can be used to model the number of hawks in the region x years after 2016.

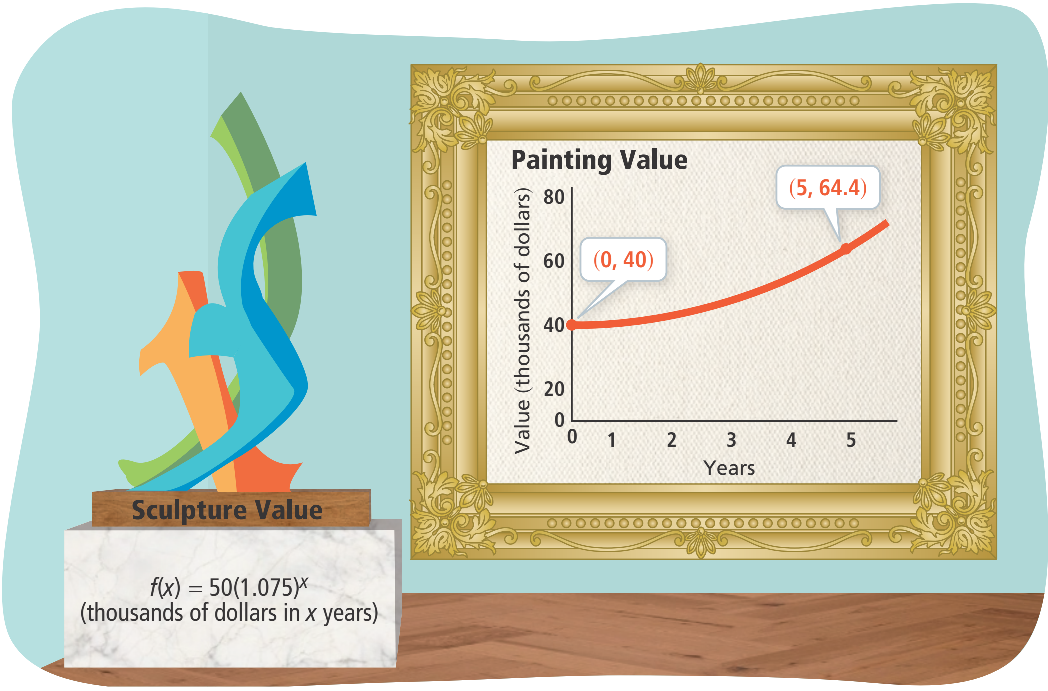
a. Is the population increasing or decreasing? Explain.

b. In what year will the number of hawks reach 280?

APPLICATION

EXAMPLE 5 Compare Two Exponential Functions

A museum purchased a painting and a sculpture in the same year. Their changing values are modeled as shown. Find the average rate of change of the value of each art work over the 5-year period. Which art work's value is increasing more quickly?



Sculpture

$f(0) = 50(1.075)^0 = 50$

$f(5) = 50(1.075)^5 \approx 71.78$

$\frac{y_2 - y_1}{x_2 - x_1} = \frac{71.78 - 50}{5 - 0} = \frac{21.78}{5} = 4.356$

The sculpture's value increased at an average of \$4,356 per year.

Painting

$\frac{y_2 - y_1}{x_2 - x_1} = \frac{64.4 - 40}{5 - 0} = \frac{24.4}{5} = 4.88$

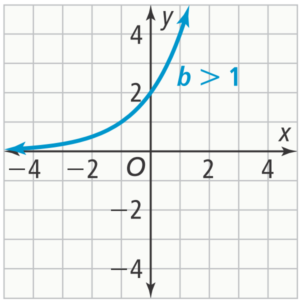
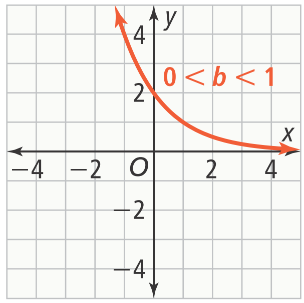
The painting's value increased at an average of \$4,880 per year.

MODEL WITH MATHEMATICS
The average rate of change of $f(x)$ from x_1 to x_2 is the slope of the line containing the points $(x_1, f(x_1))$ and $(x_2, (f(x_2))$.

Over the 5-year period, the value of the painting increased at a greater average rate than the value of the sculpture.

Try It! **5.** In Example 5, will the value of the painting ever surpass the value of the sculpture according to the models? Explain.

CONCEPT SUMMARY Key Features of Exponential Functions

	Exponential Growth	Exponential Decay
GRAPHS	 Growth factor: $1 + r$	 Decay factor: $1 - r$
EQUATIONS	$y = a \cdot b^x$, for $b > 1$	$y = a \cdot b^x$, for $0 < b < 1$
KEY FEATURES	Domain: All real numbers Range: $\{y \mid y \geq 0\}$ Intercepts: $(0, a)$ Asymptote: x-axis	Domain: All real numbers Range: $\{y \mid y \geq 0\}$ Intercepts: $(0, a)$ Asymptote: x-axis
END BEHAVIOR	As $x \to -\infty$, $y \to 0$ As $x \to \infty$, $y \to \infty$	As $x \to -\infty$, $y \to \infty$ As $x \to \infty$, $y \to 0$
MODELS	Growth: $A(t) = a(1 + r)^t$	Decay: $A(t) = a(1 - r)^t$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How do graphs and equations reveal key features of exponential growth and decay functions?

2. **Vocabulary** How do *exponential functions* differ from polynomial and rational functions?

3. **Error Analysis** Charles claimed the function $f(x) = \left(\frac{3}{2}\right)^x$ represents exponential decay. Explain the error Charles made.

4. **Communicate Precisely** How are exponential growth functions similar to exponential decay functions? How are they different?

Do You KNOW HOW?

5. Graph the function $f(x) = 4 \times 3^x$. Identify the domain, range, intercept, and asymptote, and describe the end behavior.

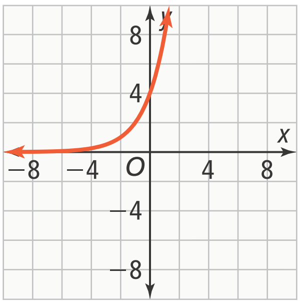
6. The exponential function $f(x) = 2500(0.4)^x$ models the amount of money in Zachary's savings account over the last 10 years. Is Zachary's account balance increasing or decreasing? Write the base in terms of the rate of growth or decay.

7. Describe how the graph of $g(x) = 4(0.5)^{x-3}$ compares to the graph of $f(x) = 4(0.5)^x$.

8. Two trucks were purchased by a landscaping company in 2016. Their values are modeled by the functions $f(x) = 35(0.85)^x$ and $g(x) = 46(0.75)^x$ where x is the number of years since 2016. Which function models the truck that is worth the most after 5 years? Explain.

UNDERSTAND

9. **Use Structure** What value of a completes the equation $y = a \cdot 2^x$ for the exponential growth function shown below?



10. **Make Sense and Persevere** Cindy found a collection of baseball cards in her attic worth $8,000. The collection is estimated to increase in value by 1.5% per year. Write an exponential growth function and find the value of the collection after 7 years.

11. **Error Analysis** Describe and correct the error a student made in identifying the growth or decay factor for the function $y = 2.55(0.7)^x$.

Step 1 The base of the function is 0.7, so it represents exponential decay.

Step 2 The function in the form $y = a(1 - r)^x$ is $y = 2.55(1 - 0.7)^x$.

Step 3 The decay factor is 0.3.

12. **Reason** In 2000, the population of Pensacola was 56,255 and it decreased to 51,923 in 2010. If this population decrease were modeled by an exponential decay function, what value would represent the y-intercept? Explain your reasoning.

13. **Mathematical Connections** Describe how the graph of $g(x) = 6 \cdot 2^{x+1} - 4$ compares to the graph of $f(x) = 6 \cdot 2^x$.

PRACTICE

Identify the domain, range, intercept, and asymptote of each exponential function. Then describe the end behavior. SEE EXAMPLE 1

14. $f(x) = 5 \cdot 3^x$

15. $f(x) = 0.75\left(\frac{2}{3}\right)^x$

16. $f(x) = 4\left(\frac{1}{2}\right)^x$

17. $f(x) = 7 \cdot 2^x$

Determine whether each function represents exponential growth or decay. Write the base in terms of the rate of growth or decay, identify *r*, and interpret the rate of growth or decay.
SEE EXAMPLES 3 AND 4

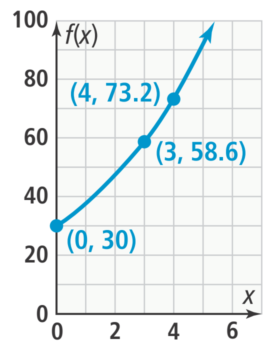
18. $y = 100 \cdot 2.5^x$

19. $f(x) = 10{,}200\left(\frac{3}{5}\right)^x$

20. $f(x) = 12{,}000\left(\frac{7}{10}\right)^x$

21. $y = 450 \cdot 2^x$

22. The function $f(x)$, shown in the graph, represents an exponential growth function. Compare the average rate of change of $f(x)$ to the average rate of change of the exponential growth function $g(x) = 25\,(1.4)^x$. Use the interval [0, 4]. SEE EXAMPLE 5



23. Write a function $g(x)$ that represents the exponential function $f(x) = 2^x$ after a vertical stretch of 6 and a reflection across the x-axis. Graph both functions. SEE EXAMPLE 2

24. The population of Port St. Joe, Florida, was 3,644 in 2000. It is expected to decrease by about 0.55% per year. Write an exponential decay function and use it to approximate the population in 2020. SEE EXAMPLE 4

APPLY

25. Model With Mathematics A colony of bacteria starts with 50 organisms and quadruples each day. Write an exponential function, $P(t)$, that represents the population of the bacteria after t days. Then find the number of bacteria that will be in the colony after 5 days.

26. Higher Order Thinking The number of teams y remaining in a single elimination tournament can be found using the exponential function $y = 128\left(\frac{1}{2}\right)^x$, where x is the number of rounds played in the tournament.

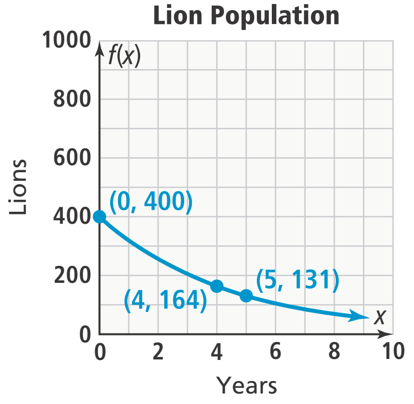
a. Determine whether the function represents exponential growth or decay. Explain.

b. What does 128 represent in the function?

c. What percent of the teams are eliminated after each round? Explain how you know.

d. Graph the function. What is a reasonable domain and range for the function? Explain.

27. Construct Arguments The function shown in the graph represents the number of lions in a region after x years, where the rate of decay is 20%. The number of zebras in that same region after x years can be modeled by the function $f(x) = 300(0.95)^x$. A representative for a conservationist group claims there will be fewer lions than zebras within 2 years. Is the representative correct? Justify your answer.



ASSESSMENT PRACTICE

28. The exponential function $g(x) = 3^{x-1} + 6$ is a transformation of the function $f(x) = 3^x$. Choose the statement that accurately describes how the graph of $g(x)$ compares to the graph of $f(x)$. F-BF.2.3

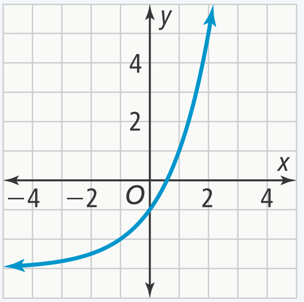
Ⓐ $g(x)$ is translated 1 unit to the left and 6 units down.

Ⓑ $g(x)$ is translated 1 unit to the left and 6 units up.

Ⓒ $g(x)$ is translated 1 unit to the right and 6 units down.

Ⓓ $g(x)$ is translated 1 unit to the right and 6 units up.

29. SAT/ACT Which of the functions defined below could be the one shown in this graph?



Ⓐ $f(x) = 4(2)^{x-1} + 3$ Ⓒ $f(x) = 4(2)^{x-1} - 3$

Ⓑ $f(x) = 4(2)^{x+1} + 3$ Ⓓ $f(x) = 4(2)^{x+1} - 3$

30. Performance Task A radioactive isotope of the element osmium Os-182 has a half-life of 21.5 hours. This means that if there are 100 grams of Os-182 in a sample, after 21.5 hours there will only be 50 grams of that isotope remaining.

Part A Write an exponential decay function to model the amount of Os-182 in a sample over time. Use A_0 for the initial amount and A for the amount after time t in hours.

Part B Use your model to predict how long it would take a sample containing 500 g of Os-182 to decay to the point where it contained only 5 g of Os-182.