Activity Assess

3-1 Graphing Polynomial Functions

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I CAN… predict the behavior of polynomial functions.

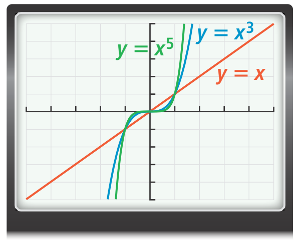
VOCABULARY

- degree of a polynomial
- leading coefficient
- polynomial function
- relative maximum
- relative minimum
- standard form of a polynomial
- turning point

MAFS.912.F-IF.2.4–For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity. Also F-IF.2.6, F-IF.3.7.c
MAFS.K12.MP.2.1, MP.5.1, MP.7.1

EXPLORE & REASON

Consider functions of the form $f(x) = x^n$, where n is a positive integer.



A. Graph $f(x) = x^n$ for $n = 1$, 3, and 5. Look at the graphs in Quadrant I. As the exponent increases, what is happening to the graphs? Which quadrants do the graphs pass through?

B. Look for Relationships Now graph $f(x) = x^n$ for $n = 2$, 4, and 6. What happens to these graphs in Quadrant I as the exponent increases? Which quadrants do the graphs pass through?

C. Write two equations in the form $f(x) = x^n$ with graphs that you predict are in Quadrants I and II. Write two equations with graphs that you predict are in Quadrants I and III. Use graphing technology to test your predictions.

ESSENTIAL QUESTION

How do the key features of a polynomial function help you sketch its graph?

EXAMPLE 1 Classify Polynomials

How can you write a polynomial in standard form and use it to identify the leading coefficient, the degree, and the number of terms?

$$-4x + 9 + 2x^3$$

Recall that a polynomial is a monomial or the sum of one or more monomials, called terms. The degree of a term with one variable is the exponent of that variable.

Degree of $-4x$: 1 Degree of 9: 0 Degree of $2x^3$: 3

Standard form of a polynomial shows any like terms combined and the terms by degree in descending numerical order.

Standard form of this polynomial is:

$$2x^3 - 4x + 9$$

The **leading coefficient** refers to the non-zero factor that is multiplied by the greatest power of x. The leading coefficient of this polynomial is 2.

The polynomial has three terms, so it is called a *trinomial*.

The **degree of a polynomial** is the greatest degree of any of the terms. This is a polynomial of degree 3, also known as a *cubic polynomial*.

USE STRUCTURE
Note that there is no x^2-term in the polynomial $2x^3 - 4x + 9$. In some cases, it may be useful to write the polynomial as $2x^3 + 0x^2 - 4x + 9$.

Try It!

1. What is each polynomial in standard form and what are the leading coefficient, the degree, and the number of terms of each?

a. $2x - 3x^4 + 6 - 5x^3$

b. $x^5 + 2x^6 - 3x^4 - 8x + 4x^3$

CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Understand End Behavior of Polynomial Functions

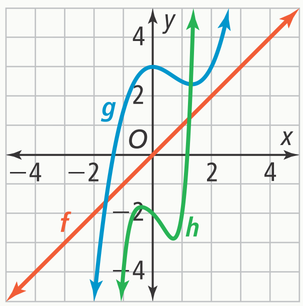
How do the sign of the leading coefficient and the degree of a polynomial affect the end behavior of the graph of a polynomial function?

A **polynomial function** is a function whose rule is a polynomial. The **end behavior** of a graph describes what happens to the function values as x approaches positive and negative infinity.

LOOK FOR RELATIONSHIPS
Though a polynomial function may have many terms, the leading term determines the end behavior because it has the greatest exponent and therefore the greatest impact on function values when x is very large or very small.

Odd Degree, Positive Leading Coefficient

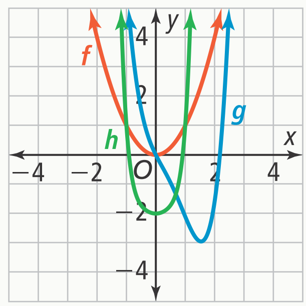
$f(x) = x$; degree 1
$g(x) = 0.5x^3 - x^2 + 3$; degree 3
$h(x) = 2x^5 - x^2 - x - 2$; degree 5



End behavior is similar to the linear parent function $f(x) = x$.

Even Degree, Positive Leading Coefficient

$f(x) = x^2$; degree 2
$g(x) = 0.9x^4 - 2x^3 + x^2 - 2x$; degree 4
$h(x) = 2x^6 + x^2 - 2$; degree 6

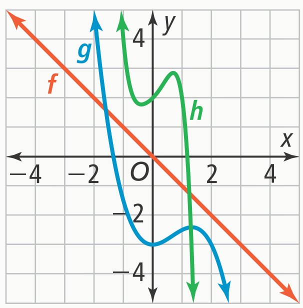


End is behavior similar to the quadratic parent function $f(x) = x^2$.

Recall that a reflection of a function across the x-axis occurs when the function is negated: $f(x)$ becomes $-f(x)$. The end behavior of a function is similarly affected when the leading coefficient is negative.

Odd Degree, Negative Leading Coefficient

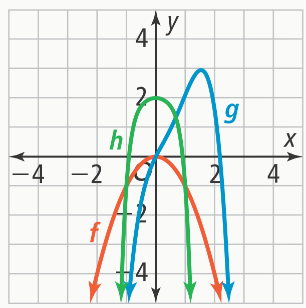
$f(x) = -x$; degree 1
$g(x) = -0.5x^3 - x^2 - 3$; degree 3
$h(x) = -2x^5 + x^2 + x + 2$; degree 5



End behavior is similar to $f(x) = -x$.

Even Degree, Negative Leading Coefficient

$f(x) = -x^2$; degree 2
$g(x) = -0.9x^4 + 2x^3 - x^2 + 2x$; degree 4
$h(x) = -2x^6 - x^2 + 2$; degree 6



End behavior is similar to $f(x) = -x^2$.

Try It! 2. Use the leading coefficient and degree of the polynomial function to determine the end behavior of each graph.

a. $f(x) = 2x^6 - 5x^5 + 6x^4 - x^3 + 4x^2 - x + 1$

b. $g(x) = -5x^3 + 8x + 4$

EXAMPLE 3 Graph a Polynomial Function

Consider the polynomial function $f(x) = -0.5x^4 + 3x^2 + 2$.

A. How can you use a table of values to identify key features and sketch a graph of the function?

Make a table of values and identify intervals where the function is increasing and decreasing.

x	$f(x)$	
–3	–11.5	increasing
–2	6	
–1	4.5	decreasing
0	2	
1	4.5	increasing
2	6	
3	–11.5	decreasing

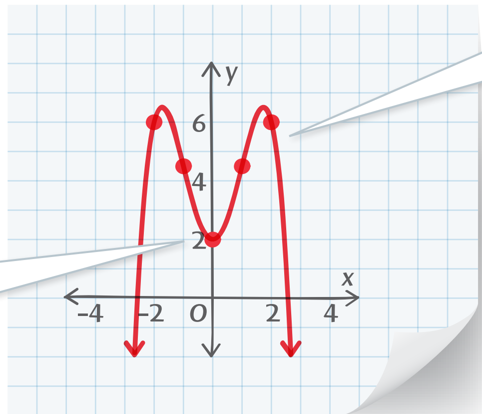
Points where the function values change from increasing to decreasing, or vice-versa, are **turning points**. This function has approximate turning points when the value of x is between –2 and –1, –1 and 0, and 1 and 2.

USE APPROPRIATE TOOLS
It can be very difficult to locate the precise turning points and zeros of polynomial functions. Graphing technology can help identify these points.

This is a polynomial function with an even degree and a negative leading coefficient, so both ends of the graph will trend toward $-\infty$.

Plot the points and sketch the graph with a smooth curve.

A point where the function has the least value over an interval is a **relative minimum**. This function has a relative minimum near (0, 2).



A point where the function has the greatest value over an interval is a **relative maximum**. This function has two relative maximums near (–2, 6) and (2, 6).

B. How can you use the graph to estimate the average rate of change over the interval [–2, 0]?

Recall that the average rate of change is $\frac{f(b) - f(a)}{b - a}$ for two points on a graph $(a, f(a))$ and $(b, f(b))$.

$$\text{Average rate of change} = \frac{f(b) - f(a)}{b - a}$$
$$= \frac{2 - 6}{0 - (-2)}$$
$$= -2$$

Substitute (–2, 6) and (0, 2).

The average rate of change over the interval [–2, 0] is –2.

Try It! 3. Consider the polynomial function $f(x) = x^5 + 18x^2 + 10x + 1$.

a. Make a table of values to identify key features and sketch a graph of the function.

b. Find the average rate of change over the interval [0, 2].

EXAMPLE 4 Sketch the Graph from a Verbal Description

How can you sketch a graph of the polynomial function *f* from a verbal description?

- ***f*(*x*) is positive on the intervals (−∞, −4) and (−1, 4).**
- ***f*(*x*) is negative on the intervals (−4, −1) and (4, ∞).**
- ***f*(*x*) is decreasing on the intervals (−∞, −2.67) and (2, ∞).**
- ***f*(*x*) is increasing on the interval (−2.67, 2).**

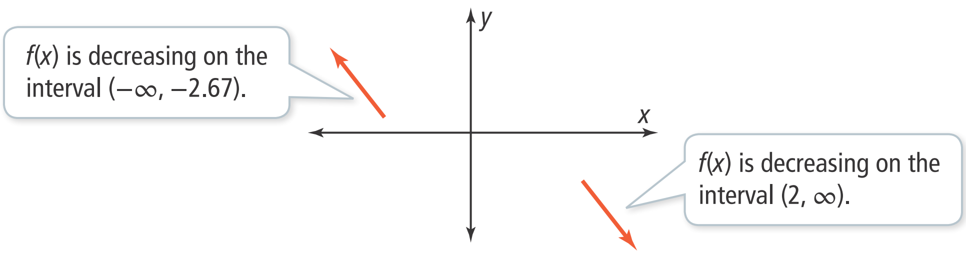
COMMON ERROR
Positive/negative behavior only tells where the function's graph lies above or below the *x*-axis. It does not indicate whether the function is increasing or decreasing.

Step 1: Identify or estimate *x*-intercepts. The function values change signs at $x = -4$, $x = -1$, and $x = 4$.

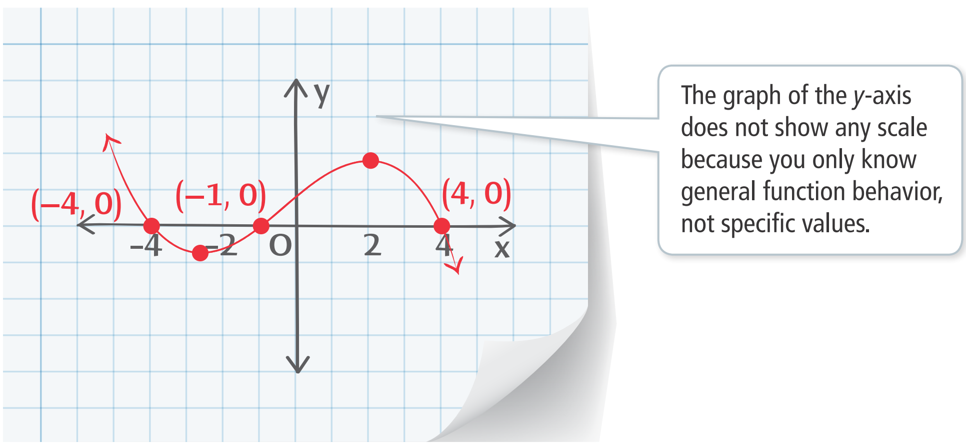
Step 2: Identify or estimate turning points. The function changes direction at $x = -2.67$ and $x = 2$.

- There is a relative minimum at $x = -2.67$.
- There is a relative maximum at $x = 2$.

Step 3: Evaluate end behavior.



Step 4: Sketch the graph.



Try It! 4. Use the information below to sketch a graph of the polynomial function $y = f(x)$.

- $f(x)$ is positive on the intervals (−2, −1) and (1, 2).
- $f(x)$ is negative on the intervals (−∞, −2), (−1, 1), and (2, ∞).
- $f(x)$ is increasing on the interval (−∞, −1.5) and (0, 1.5).
- $f(x)$ is decreasing on the intervals (−1.5, 0) and (1.5, ∞).

APPLICATION

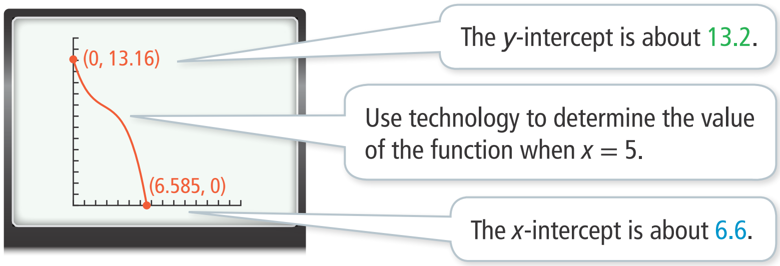
EXAMPLE 5 Interpret a Polynomial Model

In science class, Abby mixes a fixed amount of baking soda with different amounts of vinegar in a bottle capped by a balloon. She records the amount of time it takes the gases produced by the reaction to inflate the balloon.

From her data, Abby created a function to model the situation. For x quarter-cups of vinegar, it takes $t(x) = -0.12x^3 + x^2 - 3.38x + 13.16$ seconds to inflate the balloon.

A. How long would it take to inflate the balloon with 5 quarter-cups of vinegar?

Use technology to sketch the graph.



When $x = 5$, the value of the function is about 6.3. This means that if Abby uses 5 quarter-cups of vinegar, the balloon will inflate in approximately 6.3 seconds.

B. What do the x- and y-intercepts of the graph mean in this context? Do those values make sense?

The x-intercept is approximately 6.6 which means that if 6.6 cups of vinegar are used, the balloon would inflate in 0 seconds.

The y-intercept is approximately 13.2, which means that if no vinegar is used, the balloon will inflate in 13.2 seconds.

Neither the x- nor the y-intercept make sense in this context. Therefore, we must limit the domain and range when considering this model.

STUDY TIP

Recall that when you are using a graph in a real-world context, you need to consider the context when thinking about domain and range. Does it make sense for x to be negative? Does it make sense for y to be negative?

Try It!

5. Danielle is engineering a new brand of shoes. For x shoes sold, in thousands, a profit of $p(x) = -3x^4 + 4x^3 - 2x^2 + 5x + 10$ dollars, in ten thousands, will be earned.

a. How much will be earned in profit for selling 1,000 shoes?

b. What do the x- and y-intercepts of the graph mean in this context? Do those values make sense?

CONCEPT SUMMARY Graphing Polynomial Functions

WORDS

A **polynomial function** is a function whose rule is either a monomial or a sum of monomials.

KEY FEATURES

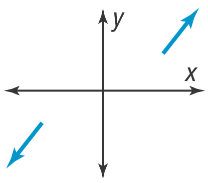
Turning points – function values change from increasing to decreasing, or vice-versa

Relative minimum – changes from decreasing to increasing

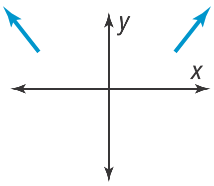
Relative maximum – changes from increasing to decreasing

GRAPHS

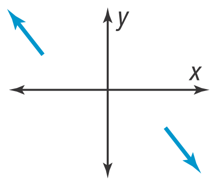
End behavior depends on the degree of the polynomial and the sign of its leading coefficient.



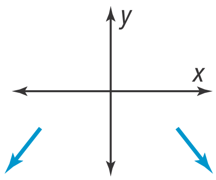
Degree: odd
Leading Coefficient: +



Degree: even
Leading Coefficient: +



Degree: odd
Leading Coefficient: –



Degree: even
Leading Coefficient: –

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How do the key features of a polynomial function help you sketch its graph?

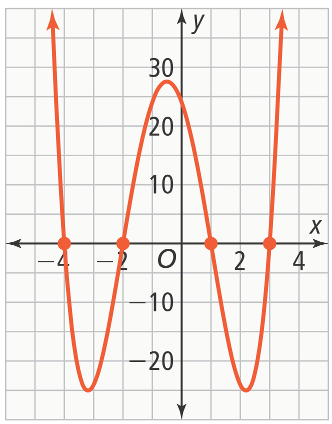
2. **Error Analysis** Allie said the degree of the polynomial function $f(x) = x^5 + 2x^4 + 3x^3 - 2x^6 - 9x^2 - 6x + 4$ is 5. Explain and correct Allie's error.

3. **Vocabulary** Explain how to determine the leading coefficient of a polynomial function.

4. **Look for Relationships** What is the relationship between the degree and leading coefficient of a polynomial function and the end behavior of the polynomial?

Do You KNOW HOW?

The graph shows the function $f(x) = x^4 + 2x^3 - 13x^2 - 14x + 24$. Find the following.



5. number of terms
6. degree
7. leading coefficient
8. end behavior
9. turning point(s)
10. x-intercept(s)
11. relative minimum(s)
12. relative maximum(s)

UNDERSTAND

13. **Make Sense and Persevere** The table shows some values of a polynomial function. Deshawn says there are turning points between the x-values -3 and -2 and between 0 and 1. He also says there is a relative minimum between the x-values -3 and -2, and a relative maximum between 0 and 1. Sketch a graph that shows how Deshawn could be correct and another graph that shows how Deshawn could be incorrect.

x	-5	-4	-3	-2	-1	0	1	2
$f(x)$	-1004	129	220	85	12	1	4	165

14. **Higher Order Thinking** Use the information below about a polynomial function in standard form to write a possible polynomial function. Explain how you determined your function and graph it to verify that it satisfies the criteria.

- 6 terms
- y-intercept at 1
- end behavior: As $x \to -\infty$, $y \to +\infty$.
  As $x \to +\infty$, $y \to -\infty$.

15. **Reason** An analyst for a new company used the first three years of revenue data to project future revenue for the company. The analyst predicts the function $f(x) = -2x^5 + 6x^4 - x^3 + 5x^2 + 6x + 50$ will give the revenue after x years. Should the CEO expect the company to be successful? Explain.

16. **Look for Relationships** Sketch a graph of each of the functions described below.

- a cubic function with one x-intercept
- a cubic function with 2 x-intercepts
- a cubic function with 3 x-intercepts

17. **Make Sense and Persevere** Compare the rate of change for the function $f(x) = x^3 - 2x^2 + x + 1$ over the intervals [0, 2] and [2, 4].

PRACTICE

Write each polynomial function in standard form. For each function, find the degree, number of terms, and leading coefficient.
SEE EXAMPLE 1

18. $f(x) = -3x^3 + 2x^5 + x + 8x^3 - 6 + x^4 - 3x^2$

19. $f(x) = 8x^2 + 10x^7 - 7x^3 - x^4$

20. $f(x) = -x^3 + 9x + 12 - x^4 + 5x^2$

Use the leading coefficient and degree of the polynomial function to determine the end behavior of the graph. SEE EXAMPLE 2

21. $f(x) = -x^5 + 2x^4 + 3x^3 + 2x^2 - 8x + 9$

22. $f(x) = 7x^4 - 4x^3 + 7x^2 + 10x - 15$

23. $f(x) = -x^6 + 7x^5 - x^4 + 2x^3 + 9x^2 - 8x - 2$

Use a table of values to estimate the intercepts and turning points of the function. Then graph the function. SEE EXAMPLE 3

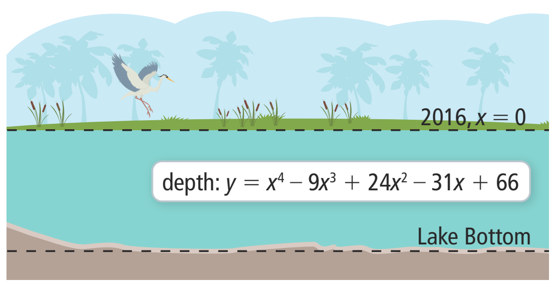
24. $f(x) = x^3 + 2x^2 - 5x - 6$

25. $f(x) = x^4 - x^3 - 21x^2 + x + 20$

26. Use the information below to sketch a graph of the polynomial function $y = f(x)$. SEE EXAMPLE 4

- $f(x)$ is positive on the intervals $(-\infty, -3)$, $(-2, 0)$, and $(2, 3)$.
- $f(x)$ is negative on the intervals $(-3, -2)$, $(0, 2)$, and $(3, \infty)$.
- $f(x)$ is increasing on the interval $(-2.67, -1)$ and $(1, 2.5)$.
- $f(x)$ is decreasing on the intervals $(-\infty, -2.67)$, $(-1, 1)$, and $(2.5, \infty)$.

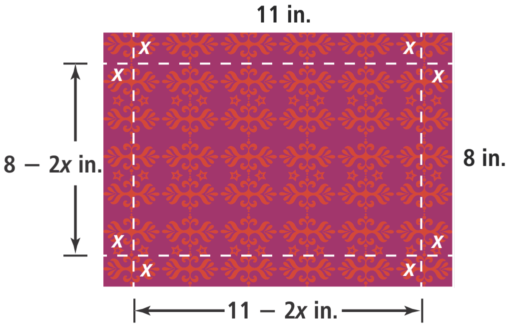
27. The equation shown models the average depth y, in feet, of a lake, x years after 2016, where $0 < x < 6$. Use technology to graph the function. In what year does this model predict a relative minimum value for the depth? SEE EXAMPLE 5



PRACTICE & PROBLEM SOLVING

APPLY

28. **Reason** Allie has a piece of construction paper that she wants to use to make an open rectangular prism. She will cut a square with side length x from each corner of the paper, so the length and width is decreased by $2x$ as shown in the diagram.



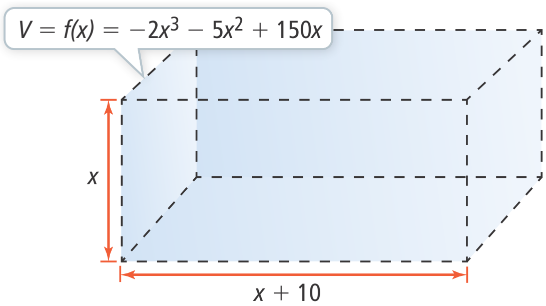
a. Write a function that models the volume of the rectangular prism.

b. Graph the function and identify a reasonable domain.

c. What do the x-intercepts of the graph mean in this context?

d. If Allie wants to maximize the volume of the box, what is the side length of the squares that should be cut from each corner of the piece of construction paper? Explain.

29. **Make Sense and Persevere** Alberto is designing a container in the shape of a rectangular prism to ship electronic devices. The length of the container is 10 inches longer than the height. The sum of the length, width, and height is 25 inches. The volume of the container, in terms of height x, is shown. Use a graphing calculator to graph the function. What do the x-intercepts of the graph mean in this context? What dimensions of the container will maximize the volume?



ASSESSMENT PRACTICE

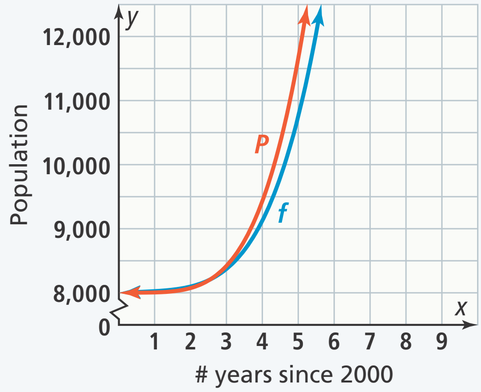
30. Graph the function $f(x) = -(x - 2)(x + 1)(x + 3)(x + 7)$. In what intervals does the function have relative maximums? Explain. F-IF.3.7.c

31. **SAT/ACT** What is the maximum number of terms a fourth-degree polynomial function in standard form can have?

Ⓐ 1 Ⓑ 2 Ⓒ 3 Ⓓ 4 Ⓔ 5

32. **Performance Task** In the year 2000, a demographer predicted the estimated population of a city, which can be modeled by the function $f(x) = 5x^4 - 4x^3 + 25x + 8{,}000$. Several years later, a statistician, using data from the U.S. Census Bureau, modeled the actual population with the function $P(x) = 7x^4 - 6x^3 + 5x + 8{,}000$. The graphs of the functions are shown.

Predicted vs. Actual Populations



Part A What is the y-intercept of each function, and what does it represent?

Part B Identify the end behaviors of f and P.

Part C Compare the average rates of change of f and P from 2003 to 2005.