1-1 Key Features of Functions

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I CAN… interpret key features of linear, quadratic, and absolute value functions given an equation or a graph.

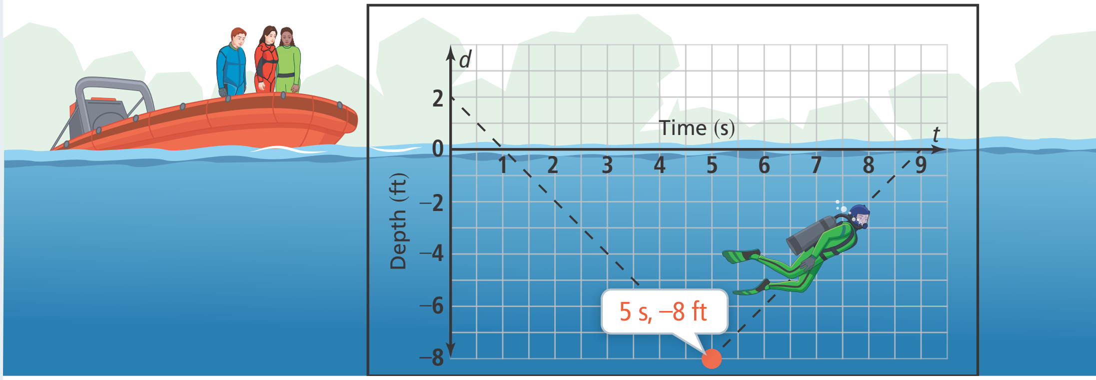
VOCABULARY

- average rate of change
- interval notation
- maximum
- minimum
- set-builder notation
- zero of a function

MAFS.912.F-BF.2.4–For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior. . .
Also F-BF.2.3, A-REI.4.11
MAFS.K12.MP.3.1, MP.4.1, MP.6.1

EXPLORE & REASON

A diver is going through ocean search-and-rescue training. The graph shows the relationship between her depth and the time in seconds since starting her dive.



A. What details can you determine about the dive from the coordinates of the point (5, –8)?

B. What is the average speed of the diver in the water? How can you tell from the graph?

C. Which point on the graph shows the starting location of the diver? Explain.

D. **Communicate Precisely** What does the V-shape of the graph tell you about the dive? What information does it not tell you about the dive?

ESSENTIAL QUESTION

How do graphs and equations reveal information about a relationship between two quantities?

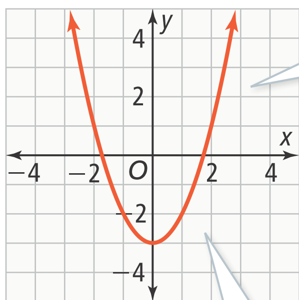
EXAMPLE 1 Understand Domain and Range

A. What are the domain and range of the function defined by $y = x^2 - 3$?

The set of all possible inputs for a relation is called the *domain*.

You can square any real number, so any number can be input for *x*.

The square of a real number is greater than or equal to 0. So the minimum value of $y = x^2 - 3$ is 0 – 3, or –3.



This graph represents a *function* because each input has exactly one output.

The set of all possible outputs for a relation is called the *range*.

There are two notations used to represent intervals of numbers like domain and range.

Set-Builder Notation uses a verbal description or an inequality to describe the numbers.

CONTINUED ON THE NEXT PAGE

CONSTRUCT ARGUMENTS Consider this explanation of the function's minimum value. How do you know that the function has no maximum value?

EXAMPLE 1 CONTINUED

Using set-builder notation, the domain of this function is $\{x \mid x \text{ is a real number}\}$.

This is read "*The set of all* x *such that* x *is a real number.*"

This is read "*The set of all* y *such that* y *is greater than or equal to* −3."

Using set-builder notation, the range of the function is $\{y \mid y \geq -3\}$.

STUDY TIP
An interval with excluded boundary points is called "open" and is represented by open circle end points on the graph. An interval with included boundary points is called "closed" and is represented by solid end points.

Interval notation represents a set of real numbers by the pair of values that are its left (minimum) and right (maximum) boundaries. Using interval notation, the domain of the function is $(-\infty, \infty)$.

Using interval notation, the range is $[-3, \infty)$.

To summarize the ways in which we can indicate intervals of numbers, refer to the table below.

Interval Notation	Words	Set Notation
$[3, 4]$	All real numbers that are greater or equal to 3 and less than or equal to 4	$\{x \mid 3 \leq x \leq 4\}$
$(3, 4]$	All real numbers that are greater than 3 and less than or equal to 4	$\{x \mid 3 < x \leq 4\}$
$[3, 4)$	All real numbers that are greater than or equal to 3 and less than 4	$\{x \mid 3 \leq x < 4\}$
$(3, 4)$	All real numbers that are greater than 3 and less than 4	$\{x \mid 3 < x < 4\}$
$[3, \infty)$	All real numbers greater than or equal to 3	$\{x \mid 3 \leq x < \infty\}$
$(-\infty, 3]$	All real numbers less than or equal to 3	$\{x \mid -\infty < x \leq 3\}$
$(-\infty, \infty)$	All real numbers	$\{x \mid -\infty < x < \infty\}$

B. An airtanker flies over forest fires and drops water at a constant rate until its tank is empty. What are the domain and range of the function that represents the volume of water the airtanker can drop in *x* seconds?

The function is $f(x) = 400x$. The airtanker cannot drop water for a negative number of seconds, so $x \geq 0$. The tanker can drop water for a maximum of $\frac{8,000}{400} = 20$ s before running out of water, so $x \leq 20$.

The domain is $\{x \mid 0 \leq x \leq 20\}$, or $[0, 20]$.

The airtanker cannot drop a negative number of gallons, and its maximum capacity is 8,000 gal.

The range is $\{y \mid 0 \leq y \leq 8,000\}$, or $[0, 8,000]$.

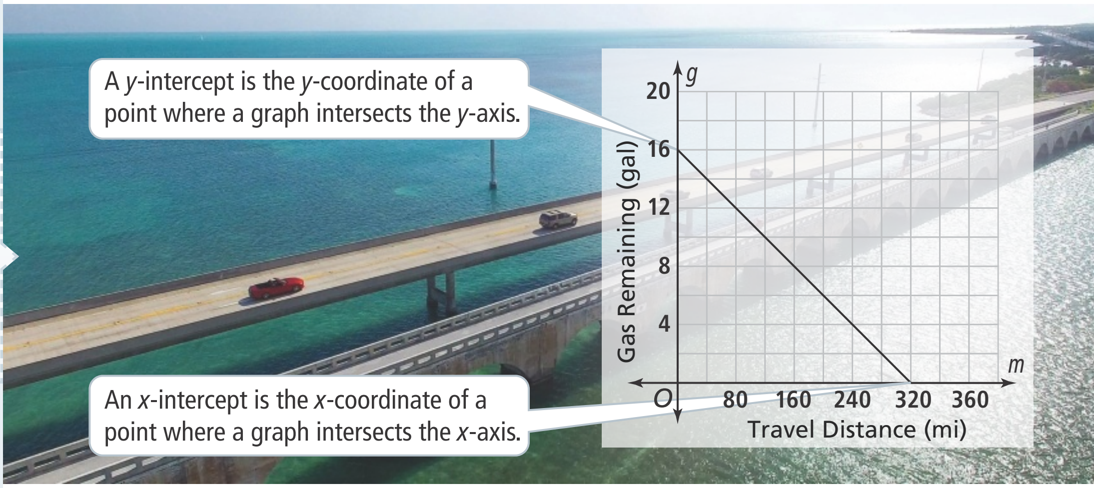
Try It! 1. What are the domain and range of each function? Write the domain and range in set-builder notation and interval notation.

a. $y = |x - 4|$ **b.** $y = 6x - 2x^2$

APPLICATION

EXAMPLE 2 Find *x*- and *y*-intercepts

A. A car starts a journey with a full tank of gas. The equation $y = 16 - 0.05x$ relates the number of gallons of gas, y, left in the tank to the number of miles the car has traveled, x. What are the x- and y-intercepts of the graph of this equation, and what do they represent about the situation?



STUDY TIP
Depending on the situation modeled by a function, the intercept(s) may not be in the domain of the function, and may not represent anything important in the situation.

The graph above intersects the x-axis at (320, 0), so the x-intercept is 320. This means that the car can travel 320 mi before it runs out of gas.

The graph intersects the y-axis at (0, 16), so the y-intercept is 16. This means the car has 16 gal of gas when it starts its trip.

B. What are the x- and y-intercepts of the graph of $y = |x| - 3$?

Find the x-intercept(s) algebraically:

$$y = |x| - 3$$
$$0 = |x| - 3$$
$$3 = |x|$$
$$x = \pm 3$$

The y-coordinate of an x-intercept is 0.

Both 3 and −3 are 3 units away from 0, so they both have an absolute value of 3.

The x-intercepts of the graph of $y = |x| - 3$ are −3 and 3. The x-intercepts are also the **zeros of the function** because they are the input values that result in a function output value of 0.

Find the y-intercept algebraically:

$$y = |x| - 3$$
$$y = |0| - 3$$
$$y = 0 - 3$$
$$y = -3$$

The x-coordinate at the y-intercept is 0.

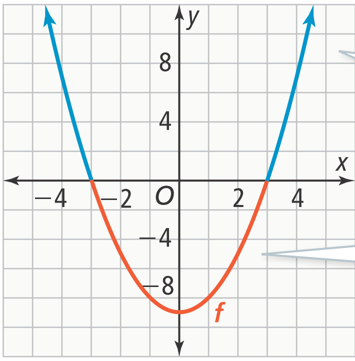
The y-intercept of the graph of $y = |x| - 3$ is −3.

Try It! **2.** What are the x- and y-intercepts of $g(x) = 4 - x^2$?

EXAMPLE 3 Identify Positive or Negative Intervals

For what intervals is $f(x) = x^2 - 9$ positive? For what intervals is the function negative?

Use technology to graph the function:



$f(x) > 0$ (above the x-axis) when $x < -3$ and when $x > 3$.

$f(x) < 0$ (below the x-axis) between the x-intercepts of −3 and 3.

COMMON ERROR
Be careful not to confuse a positive function value and a positive rate of change. A positive rate of change means the y-values of the function are increasing but are not necessarily greater than 0.

The function is positive at $(-\infty, -3)$ and $(3, \infty)$.

Parentheses indicate that a boundary point is not included.

The function is negative at $(-3, 3)$.

The function is neither positive nor negative at the x-intercepts of −3 and 3.

Try It! **3. a.** For what interval(s) is $h(x) = 2x + 10$ positive?

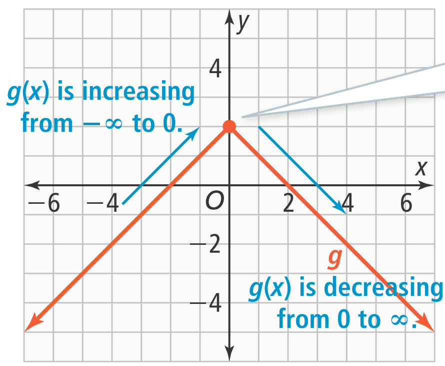
b. For what interval(s) is the function negative?

EXAMPLE 4 Identify Where a Function Increases or Decreases

For what values of x is $g(x) = 2 - |x|$ increasing? For what values is it decreasing?

Construct a table and sketch a graph to represent the function.

x	$g(x)$
−3	−1
−2	0
−1	1
0	2
1	1
2	0
3	−1



The greatest value a function attains is the ***maximum*** of the function.

The least value a function attains is the ***minimum***.

The values of $g(x)$ are increasing on the interval $(-\infty, 0)$.

The values of $g(x)$ are decreasing on the interval $(0, \infty)$.

Try It! **4.** For what values of x is each function increasing? For what values of x is it decreasing?

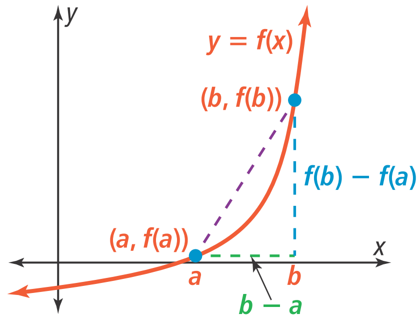
a. $f(x) = x^2 - 4x$ **b.** $f(x) = -2x - 3$

Activity Assess

CONCEPTUAL UNDERSTANDING

EXAMPLE 5 Understand Average Rate of Change Over an Interval

A. What is the average rate of change of a function $y = f(x)$ over the interval $[a, b]$?



The interval starts at a value of $f(a)$ when $x = a$ and ends at a value of $f(b)$ when $x = b$.

The total change in the function values is $f(b) - f(a)$.

The length of the interval is $b - a$.

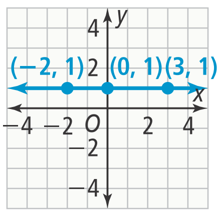
The **average rate of change** is the ratio $\frac{f(b) - f(a)}{b - a}$. This is the same as the slope of the line segment between the points $(a, f(a))$ and $(b, f(b))$.

B. What do the average rates of change over the intervals [−2, 0], [0, 3], and [−2, 3] indicate about the given functions?

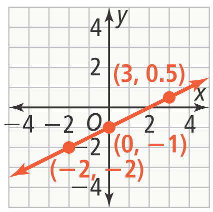
LOOK FOR RELATIONSHIPS

If (x_1, y_1) and (x_2, y_2) are points on the graph of the linear function $y = mx + b$, then the average rate of change in the interval $[x_1, x_2]$ is $m = \frac{y_2 - y_1}{x_2 - x_1}$.

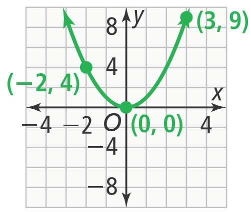
$f(x) = 1$



$g(x) = \frac{1}{2}x - 1$



$h(x) = x^2$



	$f(x) = 1$	$g(x) = \frac{1}{2}x - 1$	$h(x) = x^2$
[−2, 0]	$\frac{1 - 1}{0 - (-2)} = 0$	$\frac{-1 - (-2)}{0 - (-2)} = \frac{1}{2}$	$\frac{0 - 4}{0 - (-2)} = -2$
[0, 3]	$\frac{1 - 1}{3 - 0} = 0$	$\frac{0.5 - (-1)}{3 - 0} = \frac{1}{2}$	$\frac{9 - 0}{3 - 0} = 3$
[−2, 3]	$\frac{1 - 1}{3 - (-2)} = 0$	$\frac{0.5 - (-2)}{3 - (-2)} = \frac{1}{2}$	$\frac{9 - 4}{3 - (-2)} = 1$

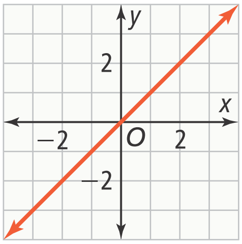
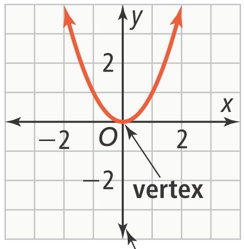
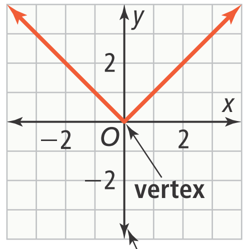
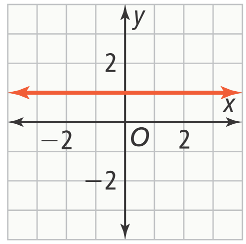
$f(x) = 1$ has the same rate of change, 0, over every interval $[a, b]$. This means it is a constant function.

$g(x) = \frac{1}{2}x - 1$ has a constant rate of change, $\frac{1}{2}$, over every interval $[a, b]$. This means it is a linear function.

$h(x) = x^2$ does not have a constant rate of change over every interval $[a, b]$. This means it is a nonlinear function.

Try It! 5. What do the average rates of change of the function $y = |x| + 2$ over the intervals [−2, 0], [0, 3], and [−2, 3] indicate about the function?

CONCEPT SUMMARY Some Functions With Key Features

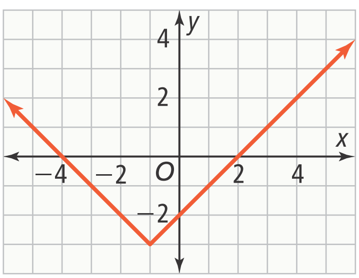
FUNCTION	Linear $y = x$	Quadratic $y = x^2$	Absolute Value $y = \|x\|$	Constant $y = 1$
GRAPH		 	 	
KEY FEATURES	Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$ Increasing: $(-\infty, \infty)$	Domain: $(-\infty, \infty)$ Range: $[0, \infty)$ Increasing: $(0, \infty)$ Decreasing: $(-\infty, 0)$	Domain: $(-\infty, \infty)$ Range: $[0, \infty)$ Increasing: $(0, \infty)$ Decreasing: $(-\infty, 0)$	Domain: $(-\infty, \infty)$ Range: $\{y \mid y = 1\}$
INTERCEPTS	The x-intercept is 0. The y-intercept is 0.	The x-intercept is 0. The y-intercept is 0.	The x-intercept is 0. The y-intercept is 0.	There is no x-intercept. The y-intercept is 1.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How do graphs and equations reveal information about a relationship between two quantities?

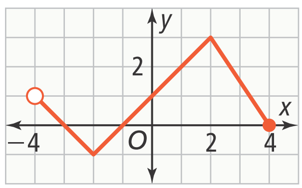
2. **Vocabulary** Define the term *zero of a function* in your own words.

3. **Error Analysis** Lonzell said the function shown in the graph is positive on the interval $(-1, 5)$ and negative on the interval $(-5, -1)$. Identify and correct Lonzell's error.



Do You KNOW HOW?

Find each key feature.



4. domain
5. range
6. x-intercept(s)
7. y-intercept(s)
8. interval(s) where the graph is positive
9. interval(s) where the graph is decreasing
10. interval(s) where the graph is increasing
11. rate of change on $[-1, 4]$

PRACTICE & PROBLEM SOLVING

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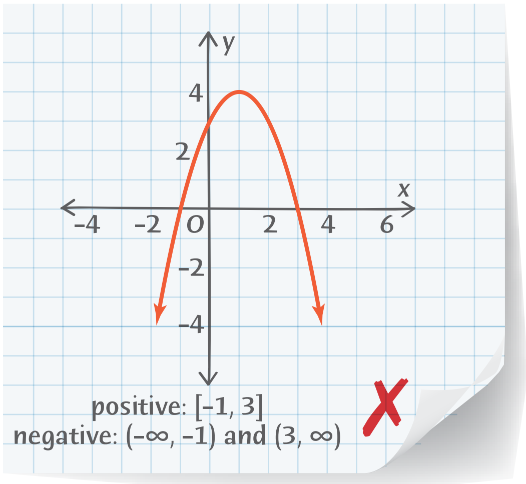
Practice Tutorial

Additional Exercises Available Online

UNDERSTAND

12. **Reason** The graph of $y = -\frac{1}{2}x + 2$ is negative over the interval $(4, \infty)$ and positive over the interval $(-\infty, 4)$. What happens on the graph when $x = 4$? Explain.

13. **Error Analysis** Describe and correct the error a student made in finding the interval(s) over which the function is positive and negative.



14. **Use Structure** Sketch a graph given the following key features.

domain: $(-4, 4)$	range: $(-4, 6]$
increasing: $(-4, 1)$	decreasing: $(1, 4)$
x-intercepts: $(-2, 0)$, $(3, 0)$	y-intercept: $(0, 4)$
negative: $(-4, -2)$ and $(3, 4)$	positive: $(-2, 3)$

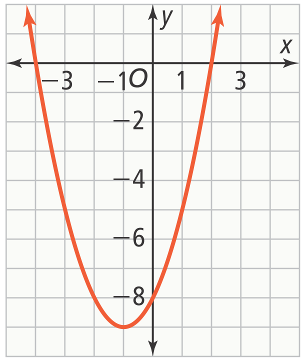
15. **Construct Arguments** A student says that all linear functions are either increasing or decreasing. Do you agree? Explain.

16. **Higher Order Thinking** A relative maximum of a function occurs at the highest point on a graph over a certain interval. A relative minimum of a function occurs at the lowest point on a graph over a certain interval. Explain how to identify a relative maximum and a relative minimum of a function using key features.

17. **Model With Mathematics** For a graph of speed in miles per hour as a function of time in hours, what does it mean when the function is increasing? Decreasing?

PRACTICE

Use the graph of the function for Exercises 18–22.



18. Identify the domain and range of the function. SEE EXAMPLE 1

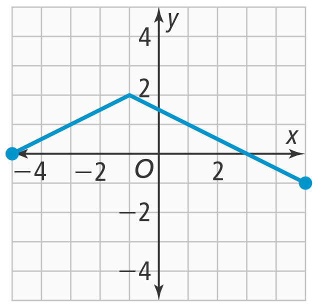
19. Identify the x- and y-intercepts of the function. SEE EXAMPLE 2

20. On what intervals is the function positive? On what intervals is it negative? SEE EXAMPLE 3

21. On what intervals is the function increasing? On what intervals is it decreasing? SEE EXAMPLE 4

22. What is the average rate of change over the interval $(-3, 2)$? SEE EXAMPLE 5

Use the graph of the function for Exercises 23–27.



23. Identify the domain and range of the function. SEE EXAMPLE 1

24. Identify the x- and y-intercepts of the function. SEE EXAMPLE 2

25. Determine over what interval the function is positive or negative. SEE EXAMPLE 3

26. Determine over what interval the function is increasing or decreasing. SEE EXAMPLE 4

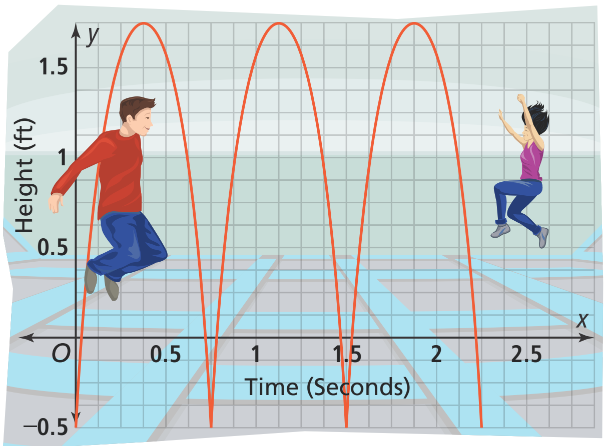
27. What is the average rate of change over the interval $(-1, 5)$? SEE EXAMPLE 5

PRACTICE & PROBLEM SOLVING

APPLY

28. Communicate Precisely Kathryn is filling an empty 100 ft^3 container with sand at a rate of 1.25 ft^3/min. Describe the key features of the graph of the amount of sand inside the container.

29. Make Sense and Persevere The graph shows a jumper's height, *y*, in feet *x* seconds after getting onto a trampoline.



a. What are the *x*- and *y*-intercepts? Explain what the *x*- and *y*-intercepts represent.

b. Over what intervals is the graph positive? Explain what the positive intervals represent.

c. Over what intervals is the graph negative? Explain what the negative intervals represent.

d. What is the average rate of change over the interval [0.75, 1.125]? Explain the meaning of the average rate of change.

30. Model With Mathematics Bailey starts playing a game on her cell phone with the battery fully charged, and plays until the phone battery dies. While playing the game, the charge in Bailey's battery decreases by half a percent per minute.

a. Write a function for the percent charge in the battery while Bailey is playing the game.

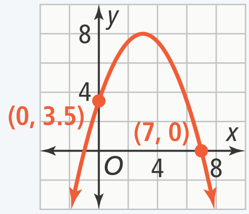
b. What is the domain and range of the function?

c. How long can Bailey play the game?

ASSESSMENT PRACTICE

31. The graph represents the path of water leaving a fountain 3.5 feet above ground. The water hits the ground 7 feet away from the fountain.

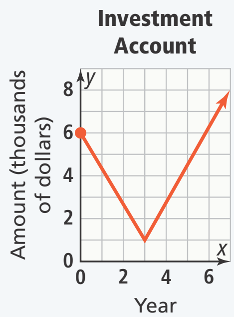
F-IF.2.4



a. What does the vertex of the graph represent?

b. If the vertex of the graph is (3, 8), on what interval is the water going up in its path? On what interval is it going down?

32. SAT/ACT The graph shows the amount of money in an investment account. Which statement is true?



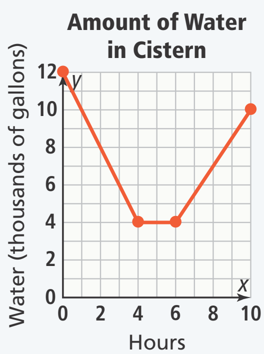
Ⓐ \$6,000 was initially invested in the account.

Ⓑ \$1,000 was initially invested in the account.

Ⓒ At Year 3, there was \$0 in the account.

Ⓓ At Year 7, there was \$0 in the account.

33. Performance Task The graph shows the amount of water in a water tank over several hours.



Part A What is the average rate of change on the interval [0, 4] and on the interval [6, 10]? What is a possible explanation for what each rate of change indicates?

Part B What is a possible explanation for what occurred between 4 and 6 h?

Part C What is the average rate of change on the interval [0, 10]? What does the rate of change mean? Does this rate of change give a good indication as to what is happening with the water in the cistern from 0 h to 10 h? Explain.