Activity | Assess

# 6-3 Logarithms

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**I CAN…** evaluate and simplify logarithms.

## VOCABULARY

- common logarithm
- logarithm
- logarithmic function
- natural logarithm

MAFS.912.F-LE.1.4–For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where *a*, *c*, and *d* are numbers and the base *b* is 2, 10, or *e*; evaluate the logarithm using technology. Also F-BF.2.4.a
MAFS.K12.MP.2.1, MP.4.1, MP.7.1

## CRITIQUE & EXPLAIN

**Earthquakes make seismic waves through the ground. The equation $y = 10^x$ relates the height, or amplitude, in microns, of a seismic wave, *y*, and the power, or magnitude, *x*, of the ground-shaking it can cause.**

| Magnitude, *x* | Amplitude, *y* |
|---|---|
| 2 | 100 |
| 3 | 1,000 |
| **?** | 5,500 |
| 4 | 10,000 |

Taylor and Chen used different methods to find the magnitude of the earthquake with amplitude 5,500.

Taylor
5,500 is halfway between 1,000 and 10,000.

3.5 is halfway between 3 and 4.

The magnitude is about 3.5.

Chen
$y = 10^x$

$10^3 = 1{,}000$
$10^4 = 10{,}000$
$10^{3.5} \approx 3{,}162$
$10^{3.7} \approx 5{,}012$
$10^{3.8} \approx 6{,}310$
$10^{3.74} \approx 5{,}500$
The magnitude is about 3.74.

**A.** What is the magnitude of an earthquake with amplitude 100,000? How do you know?

**B.** Construct Arguments Critique Taylor's and Chen's work. Is each method valid? Could either method be improved?

**C.** Describe how to express the exact value of the desired magnitude.

## ESSENTIAL QUESTION

**What are logarithms and how are they evaluated?**

CONCEPTUAL UNDERSTANDING

## EXAMPLE 1 Understand Logarithms

**Solve the equations $2x = 8$ and $2^x = 8$.**

You can use inverse operations to solve the first equation.

$$\frac{2x}{2} = \frac{8}{2}$$
$$x = 4$$

Division is the inverse of multiplication, so you can divide both sides by 2 to solve the equation.

The operation in $2^x = 8$ is exponentiation. To solve this equation, you need an inverse for exponentiation that answers the question, "To what exponent would you raise the base 2 to get 8?"

The inverse of exponentiation is called a *logarithm*. To solve the equation $2^x = 8$, you can write $\log_2 8 = x$. Solving this gives $\log_2 8 = 3$ because $2^3 = 8$.

This is read "logarithm base 2 of 8" or "log base 2 of 8."

**USE STRUCTURE**
Creating the notation $\log_2 x$ to represent the exponent to which you raise 2 to get *x* is similar to creating the radical notation $\sqrt{x}$ to represent one number you can square to get *x*.

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Activity Assess

EXAMPLE 1 CONTINUED

The **logarithm** base $b$ of $x$ is defined as follows.

$$\log_b x = y \text{ if and only if } b^y = x, \text{ for } b > 0,\ b \neq 1, \text{ and } x > 0.$$

The **logarithmic function** $y = \log_b x$ is the inverse of the exponential function $y = b^x$.

**Try It!** **1.** Write the logarithmic form of $y = 8^x$.

## CONCEPT Exponential and Logarithmic Forms

**Exponential form** shows that a base raised to an exponent equals the result.

$$a^b = c$$

**Logarithmic form** shows that the log of the result with the given base equals the exponent.

$$\log_a c = b$$

When written in logarithmic form, the number that was the result of the exponential equation is often called the argument.

## EXAMPLE 2 Convert Between Exponential and Logarithmic Forms

**STUDY TIP**
Do you remember writing *fact families* for related operations like addition and subtraction? Think of exponential and logarithmic forms as a *fact family* for the three numbers given.

**A. What is the logarithmic form of $3^4 = 81$?**

The base is 3, the exponent is 4, and the result is 81.

So, in logarithmic form,

$3^4 = 81 \rightarrow \log_3 81 = 4$.

The logarithmic form of $3^4 = 81$ is $\log_3 81 = 4$.

**B. What is the exponential form of $\log_{10} 1{,}000 = 3$?**

The base is 10, the exponent is 3, and the result (or argument) is 1,000.

So, in exponential form,

$\log_{10} 1{,}000 = 3 \rightarrow 10^3 = 1{,}000$.

The exponential form of $\log_{10} 1{,}000 = 3$ is $10^3 = 1{,}000$.

**Try It!** **2. a.** What is the logarithmic form of $7^3 = 343$?

**b.** What is the exponential form of $\log_4 16 = 2$?

## EXAMPLE 3 Evaluate Logarithms

**What is the value of each logarithmic expression?**

| | |
|---|---|
| **A.** $\log_5 125$<br>THINK: $5^? = 125$<br>Since $5^3 = 125$, $\log_5 125 = 3$. | **B.** $\log_{\frac{1}{4}} 16$<br>THINK: $\left(\frac{1}{4}\right)^? = 16$<br>Since $\left(\frac{1}{4}\right)^{-2} = 16$, $\log_{\frac{1}{4}} 16 = -2$. |
| **C.** $\log_3 0$<br>THINK: $3^? = 0$<br>There is no such power, so $\log_3 0$ is undefined. | **D.** $\log_2 2^8$<br>THINK: $2^? = 2^8$<br>Since $2^8 = 2^8$, $\log_2 2^8 = 8$. |

**GENERALIZE**
The output of any exponential function of the form $y = b^x$, with $b > 0$, is always a positive number. Therefore, the input of a logarithmic function must also be a positive number.

**Try It!** **3.** What is the value of each logarithmic expression?

**a.** $\log_3\left(\frac{1}{81}\right)$ **b.** $\log_7(-7)$ **c.** $\log_5 5^9$

## CONCEPT Common Logarithms and Natural Logarithms

The base 10 logarithm is called the **common logarithm** and is written as log $x$ with the base of 10 implied.

The base $e$ logarithm is called the **natural logarithm** and is written as ln $x$.

The expressions $\log_{10} x$ and $\log x$ mean the same thing, as do $\ln_e x$ and $\ln x$.

## EXAMPLE 4 Evaluate Common and Natural Logarithms

**What is the value of each logarithmic expression to the nearest ten-thousandth?**

**STUDY TIP**
Most calculators have keys for the common logarithm (LOG) and the natural logarithm (LN).

Check by writing the expression in exponential form and evaluating.

**A. log 900**

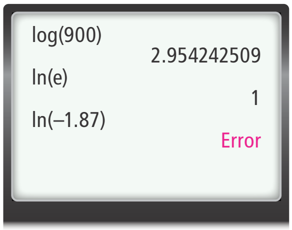
$\log 900 \approx 2.9542$ $10^{2.9542} \approx 900$

**B. ln *e***

$\ln e = 1$ $e^1 = e$

**C. ln (–1.87)**

$\ln(-1.87)$ $e^? = -1.87$




There is no exponent to which $e$ can be raised in order to get a negative number, so $\ln(-1.87)$ is undefined.

**Try It!** **4.** What is the value of each logarithmic expression to the nearest ten-thousandth?

a. log 321 b. ln 1,215 c. log 0.17

## EXAMPLE 5 Solve Equations With Logarithms

**COMMON ERROR**
Remember that 10 is not a coefficient, but a base. You cannot divide both sides by 10 and then add 1 to solve for $x$.

**What is the solution to each equation? Round to the nearest thousandth.**

**A.** $25 = 10^{x-1}$

$25 = 10^{x-1}$

$\log 25 = x - 1$ ········ Convert to logarithmic form.

$1 + \log 25 = x$ ········ Addition Property

$2.398 \approx x$ ········ Use calculator to evaluate.

**B.** $\ln(2x + 3) = 4$

$\ln(2x + 3) = 4$

$2x + 3 = e^4$ ········ Convert to exponential form.

$2x + 3 \approx 54.598$ ········ Use calculator to evaluate.

$2x \approx 51.598$ ········ Addition Property

$x \approx 25.799$ ········ Multiplication Property

**Try It!** **5.** Solve each equation. Round to the nearest thousandth.

**a.** $\log(3x - 2) = 2$ **b.** $e^{x+2} = 8$

APPLICATION

## EXAMPLE 6 Use Logarithms to Solve Problems

**The seismic energy, $x$, in joules can be estimated based on the magnitude, $m$, of an earthquake by the formula $x = 10^{1.5m+12}$. What is the magnitude of an earthquake with a seismic energy of $4.2 \times 10^{20}$ joules?**


Formulate ◀ Substitute $4.2 \times 10^{20}$ for $x$ in the formula.

$4.2 \times 10^{20} = 10^{1.5m+12}$

Compute ◀ Solve the equation for $m$.

$4.2 \times 10^{20} = 10^{1.5m+12}$ ········ Write the original equation.

$\log(4.2 \times 10^{20}) = 1.5m + 12$ ········ Write the equation in logarithmic form.

$20.6 \approx 1.5m + 12$ ········ Evaluate the logarithm.

$5.75 \approx m$ ········ Solve for $m$.

Interpret ◀ The magnitude of the earthquake is about 5.75. Verify the answer: $10^{1.5(5.75)+12} \approx 4.2 \times 10^{20}$

**Try It!** **6.** What is the magnitude of an earthquake with a seismic energy of $1.8 \times 10^{23}$ joules?

## CONCEPT SUMMARY Logarithms

| | Exponential Form | | Logarithmic Form |
|---|---|---|---|
| ALGEBRA | $b^x = y$ | $\longleftrightarrow$ | $\log_b y = x$ |
| WORDS | The base raised to the exponent is equal to a result. | | The logarithm with a base b of the result (or argument) is equal to the exponent. |
| NUMBERS | $3^4 = 81$ | $\longleftrightarrow$ | $\log_3 81 = 4$ |

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What are logarithms and how are they evaluated?

2. **Error Analysis** Amir said the expression $\log_5(-25)$ simplifies to $-2$. Explain Amir's possible error.

3. **Vocabulary** Explain the difference between the common logarithm and the natural logarithm.

4. **Make Sense and Persevere** How can logarithms help to solve an equation such as $10^t = 656$?

## Do You KNOW HOW?

**Write each equation in logarithmic form.**

**5.** $2^{-6} = \frac{1}{64}$

**6.** $e^4 \approx 54.6$

**Write each equation in exponential form.**

**7.** $\log 200 \approx 2.301$

**8.** $\ln 25 \approx 3.22$

**Evaluate the expression.**

**9.** $\log_4 64$

**10.** $\log \frac{1}{100}$

**11.** $\ln e^5$

**12.** Solve for $x$. $4e^x = 7$.

## PRACTICE & PROBLEM SOLVING

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### UNDERSTAND

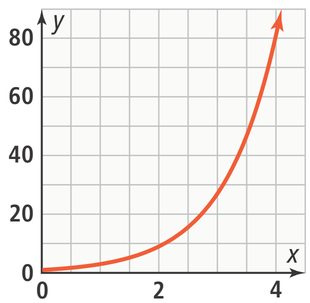
13. **Make Sense and Persevere** If the LN button on your calculator were broken, how could you still use your calculator to find the value of the expression ln 65?

14. **Error Analysis** Describe and correct the error a student made in solving an exponential equation.

$$16e^t = 98$$
$$e^t = 6.125$$
$$6.125t = \ln e$$
$$t = \frac{\ln e}{6.125}$$

✗

15. **Higher Order Thinking** Use the graph of $y = 3^x$ to estimate the value of $\log_3 50$. Explain your reasoning.




16. **Generalize** For what values of $x$ is the expression $\log_4 x < 0$ true?

17. **Use Structure** A student says that $\log_3\left(\frac{1}{27}\right)$ simplifies to $-3$. Is the student correct? Explain.

18. **Use Structure** Explain why the expression ln 1,000 is not equal to 3.

### PRACTICE

**Write the inverse of each exponential function.**
SEE EXAMPLE 1

**19.** $y = 4^x$ **20.** $y = 10^x$

**21.** $y = 7^x$ **22.** $y = a^x$

**Write each equation in logarithmic form.**
SEE EXAMPLE 2

**23.** $3^8 = 6{,}561$ **24.** $e^{-3} \approx 0.0498$

**25.** $5^0 = 1$ **26.** $7^3 = 343$

**Write each equation in exponential form.**
SEE EXAMPLE 2

**27.** $\log \frac{1}{100} = -2$ **28.** $\log_8 64 = 2$

**29.** $\ln 148.41 \approx 5$ **30.** $\log_2 \frac{1}{32} = -5$

**Evaluate each logarithmic expression.** SEE EXAMPLE 3

**31.** $\log_5 \frac{1}{125}$ **32.** $\log_6(-216)$

**33.** $\log_3 3^4$ **34.** $\log_2 32$

**35.** $\log_9 729$ **36.** $\log_8 \frac{1}{64}$

**37.** $\log_7 0$ **38.** $\log_7 7^a$

**Use a calculator to evaluate each expression. Round to the nearest ten-thousandth.** SEE EXAMPLE 4

**39.** $\log 78.5$ **40.** $\log 0.24$

**41.** $\ln(-37)$ **42.** $\ln 41.5$

**43.** $\log 12$ **44.** $\ln 3$

**Solve each equation. Round answers to the nearest ten-thousandth.** SEE EXAMPLES 5 AND 6

**45.** $\log(7x + 6) = 3$ **46.** $2.75e^t = 38.6$

**47.** $\ln(3x - 1) = 2$ **48.** $10^{t+1} = 50$

**49.** $1.5e^t = 27$ **50.** $\log(x - 3) = -1$

**51.** How long does it take for \$250 to grow to \$600 at 4% annual percentage rate compounded continuously? Round to the nearest year.

## PRACTICE & PROBLEM SOLVING

### APPLY

**52. Model with Mathematics** Michael invests \$1,000 in an account that earns a 4.75% annual percentage rate compounded continuously. Peter invests \$1,200 in an account that earns a 4.25% annual percentage rate compounded continuously. Which person's account will grow to \$1,800 first?

**53. Reason** The Richter magnitude of an earthquake is $R = 0.67\log(0.37E) + 1.46$, where $E$ is the energy (in kilowatt-hours) released by the earthquake.

**a.** What is the magnitude of an earthquake that releases 11,800,000,000 kilowatt-hours of energy? Round to the nearest tenth.

**b.** How many kilowatt-hours of energy would an earthquake have to release in order to be an 8.2 on the Richter scale? Round to the nearest whole number.

**c.** What number of kilowatt-hours of energy would an earthquake have to release in order for walls to crack? Round to the nearest whole number.


**54. Reason** The function $c(t) = 108e^{-0.08t} + 75$ calculates the temperature, in degrees Fahrenheit, of a cup of coffee that was handed out a drive-thru window $t$ minutes ago.

**a.** What is the temperature of the coffee in the instant that it is handed out the window?

**b.** After how many minutes is the coffee in the cup 98 degrees Fahrenheit? Round to the nearest whole minute.

### ASSESSMENT PRACTICE

**55.** Sandra invests \$500 in an account that earns a 2.5% annual percentage rate compounded continuously. How long will it take for her account to grow to \$700? F-LE.1.4

**56. SAT/ACT** In the equation $\log_3 a = b$, if $b$ is a whole number, which of the following CANNOT be a value for $a$?

Ⓐ 1 Ⓑ 3 Ⓒ 6 Ⓓ 9 Ⓔ 81

**57. Performance Task** Money is deposited into two separate accounts. The money in one account is compounded continuously. The money in the other account is not compounded continuously. Neither account has any money withdrawn in the first 6 years.

| Year | Account 1 Balance (\$) | Account 2 Balance (\$) |
|---|---|---|
| 0 | 400 | 500 |
| 1 | 433.31 | 575 |
| 2 | 469.40 | 650 |
| 3 | 508.50 | 725 |
| 4 | 550.85 | 800 |
| 5 | 596.72 | 875 |

**Part A** Write a function to calculate the amount of money in each account given $t$, the number of years since the account was opened. Describe the growth in each account.

**Part B** Will the amount of money in Account 1 ever exceed the amount of money in Account 2? Explain. If so, when will that occur?