Activity Assess

6-2 Exponential Models

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I CAN… write exponential models in different ways to solve problems.

VOCABULARY

- compound interest formula
- continuously compounded interest formula
- natural base e

MAFS.912.A-SSE.1.1.b–Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P. Also A-SSE.2.3.c, F-IF.3.8.b, F-LE.2.5
MAFS.K12.MP.1.1, MP.4.1, MP.7.1

EXPLORE & REASON

Juan is studying exponential growth of bacteria cultures. Each is carefully controlled to maintain a specific growth rate. Copy and complete the table to find the number of bacteria cells in each culture.

Culture	Initial Number of Bacteria	Growth Rate per Day	Time (days)	Final Number of Bacteria
A	10,000	8%	1	
B	10,000	4%	2	
C	10,000	2%	4	
D	10,000	1%	8	

A. What is the relationship between the daily growth rate and the time in days for each culture?

B. **Look for Relationships** Would you expect a culture with a growth rate of $\frac{1}{2}$% and a time of 16 days to have more or fewer cells than the others in the table? Explain.

ESSENTIAL QUESTION

How can you develop exponential models to represent and interpret situations?

EXAMPLE 1 Rewrite an Exponential Function to Identify a Rate

In 2015, the population of a small town was 8,000. The population is increasing at a rate of 2.5% per year. Rewrite an exponential growth function to find the monthly growth rate.

Write an exponential growth function using the annual rate to model the town's population y, in t years after 2015.

initial population; annual growth rate; years after 2015

$$y = 8{,}000(1 + 0.025)^t$$

$$y = 8{,}000(1.025)^t$$

To identify the monthly growth rate, you need the exponent to be the number of months in t years, or $12t$.

$$y = 8{,}000(1.025)^{\frac{12t}{12}}$$

Multiply the exponent by $\frac{12}{12}$ so that $12t$ represents the number of months.

$$y = 8{,}000\left(1.025^{\frac{1}{12}}\right)^{12t}$$

$$y \approx 8{,}000(1.00206)^{12t}$$

Applying the Power of a Power rule helps to reveal the monthly growth rate by producing an expression with the exponent $12t$.

The monthly growth rate is about $1.00206 - 1 = 0.00206$. The population is increasing about 0.206% per month.

COMMON ERROR
Dividing the annual growth rate by 12 does not give the exact monthly growth rate. This Example shows how to find an expression for the exact monthly rate: $1.025^{\frac{1}{12}} - 1$.

Try It! 1. The population in a small town is increasing annually by 1.8%. What is the quarterly rate of population increase?

CONCEPT Compound Interest

When interest is paid monthly, the interest earned after the first month becomes part of the new principal for the second month, and so on. Interest is earned on interest already earned. This is compound interest.

The **compound interest formula** is an exponential model that is used to calculate the value of an investment when interest is compounded.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

P = the initial principal invested

r = annual interest rate, written as a decimal

n = number of compounding periods per year

A = the value of the account after t years

EXAMPLE 2 Understand Compound Interest

Tamira invests $5,000 in an account that pays 4% annual interest. How much will there be in the account after 3 years if the interest is compounded annually, semi-annually, quarterly, or monthly?

Use the Compound Interest formula to find the amount in Tamira's account after 3 years.

	Compound Interest Formula	Amount After 3 Years ($)
Annually	$A = 5000\left(1 + \frac{0.04}{1}\right)^{3(1)}$	5,624.32
Semi-Annually	$A = 5000\left(1 + \frac{0.04}{2}\right)^{3(2)}$	5,630.81
Quarterly	$A = 5000\left(1 + \frac{0.04}{4}\right)^{3(4)}$	5,634.13
Monthly	$A = 5000\left(1 + \frac{0.04}{12}\right)^{3(12)}$	5,636.36

REASON
The more frequently interest is added to the account, the earlier that interest generates more interest. This reasoning supports the trend shown in the table.

As the number of compounding periods increases, the amount in the account also increases.

Try It! 2. $3,000 is invested in an account that earns 3% annual interest, compounded monthly.

a. What is the value of the account after 10 years?

b. What is the value of the account after 100 years?

CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Understanding Continuously Compounded Interest

Consider an investment of \$1 in an account that pays a 100% annual interest rate for one year. The equation $A = 1\left(1 + \frac{1}{n}\right)^{n(1)} = \left(1 + \frac{1}{n}\right)^{n}$ gives the amount in the account after one year for the number of compounding periods n. Find the value of the account for the number of periods given in the table.

Number of Periods, n	Value of $\left(1 + \frac{1}{n}\right)^{n}$
1	$\left(1 + \frac{1}{1}\right)^{1} = 2$
10	$\left(1 + \frac{1}{10}\right)^{10} = 2.59374246$
100	$\left(1 + \frac{1}{100}\right)^{100} = 2.704813829$
1000	$\left(1 + \frac{1}{1,000}\right)^{1,000} = 2.716923932$
10000	$\left(1 + \frac{1}{10,000}\right)^{10,000} = 2.718145927$
100000	$\left(1 + \frac{1}{100,000}\right)^{100,000} = 2.718268237$

Notice that as n continues to increase, the value of the account remains very close to 2.718. This special number is called the *natural base.*

The **natural base e** is defined as the value that the expression $\left(1 + \frac{1}{x}\right)^{x}$ approaches as $x \to +\infty$. The number e is an irrational number.

$$e = 2.718281828459\ldots$$

The number e is the base in the **continuously compounded interest formula**.

$A = Pe^{rt}$

P = the initial principal invested
e = the natural base
r = annual interest rate, written as a decimal
A = the value of the account after t years

Try It! **3.** If you continued the table for $n = 1,000,000$, would the value in the account increase or decrease? How do you know?

EXAMPLE 4 Find Continuously Compounded Interest

Regina invests \$12,600 in an account that earns 3.2% annual interest, compounded continuously. What is the value of the account after 12 years? Round your answer to the nearest dollar.

Use the continuously compounded interest formula with $P = 12,600$, $r = 0.032$, and $t = 12$.

$$\begin{aligned} A &= Pe^{rt} \\ &= 12,600e^{0.032(12)} \\ &= 12,600e^{0.384} \\ &\approx 18,498.63 \end{aligned}$$

To evaluate $e^{0.384}$, use the e^x key on your calculator.

To the nearest dollar, the value of the account after 12 years is \$18,499.

CONTINUED ON THE NEXT PAGE

COMMON ERROR
Be sure that when you evaluate $e^{0.032(12)}$ you either simplify 0.032(12) as 0.384 first, or use parentheses to ensure that e is raised to the entire product, rather than just the first factor.

Try It! **4.** You invest $125,000 in an account that earns 4.75% annual interest, compounded continuously.

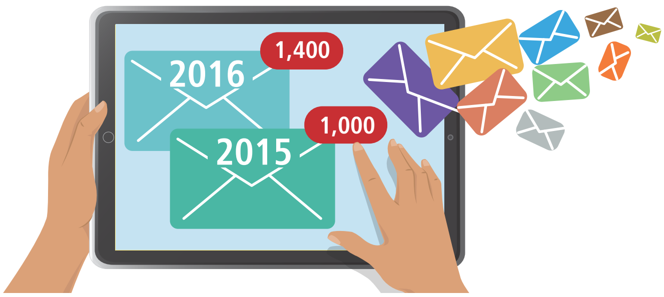
a. What is the value of the account after 15 years?

b. What is the value of the account after 30 years?

APPLICATION

EXAMPLE 5 Use Two Points to Find an Exponential Model

Tia knew that the number of e-mails she sent was growing exponentially. She generated a record of the number of e-mails she sent each year since 2009. What is an exponential model that describes the data?



Write an exponential model in the form $y = a \cdot b^x$, with y equal to the number of e-mails in hundreds and x equal to the number of years since 2009. Use the data to find the values of the constants a and b.

COMMON ERROR
Remember that the growth factor $(1 + r)$ is different from the growth rate (r). In this example, the growth factor is 1.4 while the growth rate is 0.4, or 40%.

The growth factor for Tia's e-mails in the two consecutive years was $\frac{14}{10}$, or 1.4.

When data points have consecutive x-values, the growth factor, b, is the ratio of their y-values.

Use the value of b and one of the data points to find the initial value, a.

$y = a \cdot b^x$ ········ Write an exponential growth equation.

$14 = a(1.4)^7$ ········ Substitute 1.4 for b, 7 for x, and 14 for y.

$\frac{14}{(1.4)^7} = a$ ········ Division Property of Equality

$1.33 \approx a$ ········ Simplify.

So, the function $y = 1.33(1.4)^x$ models the number of e-mails (in hundreds) Tia sends x years after 2009.

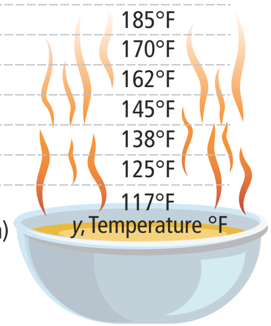
Try It! **5.** A surveyor determined the value of an area of land over a period of several years since 1950. The land was worth $31,000 in 1954 and $35,000 in 1955. Use the data to determine an exponential model that describes the value of the land.

APPLICATION

EXAMPLE 6 Use Regression to Find an Exponential Model

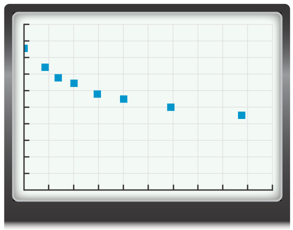
Randy is making soup. The soup reaches the boiling point and then, as shown by the data, begins to cool off. Randy wants to serve the soup when it is about 80°F, or about 10 degrees above room temperature (68°F).

x, time (min)	y, Temperature °F
00:00	212°F
02:00	185°F
03:30	170°F
05:00	162°F
07:30	145°F
10:00	138°F
15:00	125°F
20:00	117°F



A. Explain why the temperature might follow an exponential decay curve as it approaches room temperature.

A scatter plot of the data shows the soup cooling toward room temperature. The graph is not a line.



The rate of cooling appears to slow as the graph approaches room temperature, around 68° F. This indicates exponential decay toward an asymptote of $y = 68$.

B. Find an exponential model for the data. Use your model to determine when Randy should serve the soup.

Step 1 Enter the data as lists in a graphing calculator. Because the temperature values approach 68°F, subtract 68 from each temperature value.

Most graphing calculators will only calculate exponential regressions for data values that approach 0. You can subtract 68 so your data will approach 0. Then you can undo the adjustment in **Step 3** below.

L1	L2	L3 2
2	117	
3.5	102	
5	94	
7.5	77	
10	70	
15	57	
20	49	

L2(8)=49

Step 2 Use the calculator to find an exponential regression equation. The exponential model that best fits the data is $y = 126.35(0.9492)^x$.

Step 3 Translate this function up vertically by 68 units.

The translated model is $y = 126.35(0.9492)^x + 68$.

Use the translated model to find when the soup has a temperature of about 80°F.

X	Y1
42	82.142
43	82.424
44	80.742
45	80.094
46	79.48
47	78.897
48	78.343

X=45

The soup has a temperature of about 80°F after 45 minutes.

So, Randy should serve the soup about 45 minutes after it begins to cool.

STUDY TIP
The procedure for determining an exponential regression model for data may be slightly different on your graphing calculator, but the steps should be very similar.

Try It! 6. According to the model in Example 6, what was the approximate temperature 35 minutes after cooling started?

CONCEPT SUMMARY Writing Exponential Models

	General Exponential Model	Compound Interest	Continuously Compounded Interest
ALGEBRA	$y = a \cdot b^x$	$A = P\left(1 + \frac{r}{n}\right)^{nt}$	$A = Pe^{rt}$
NUMBERS	A necklace costs \$250 and increases in value by 2% per year. $a =$ initial amount \$250 $b =$ growth factor 1.02 $x =$ number of years $y = 250(1.02)^x$	A principal of \$3,000 is invested at 5% annual interest, compounded monthly, for 4 years. $P = 3{,}000$ $r = 5\%$ $n = 12$ compounding periods per year $t = 4$ years $A = 3000\left(1 + \frac{0.05}{12}\right)^{(12)(4)}$	A principal of \$3,000 is invested at 5% continuously compounded interest for 4 years. $P = 3{,}000$ $r = 5\%$ $t = 4$ years $A = 3000e^{(0.05)(4)}$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** Why do you develop exponential models to represent and interpret situations?

2. **Error Analysis** The exponential model $y = 5{,}000(1.05)^t$ represents the amount Yori earns in an account after t years when \$5,000 is invested. Yori said the monthly interest rate of the exponential model is 5%. Explain Yori's error.

3. **Vocabulary** Explain the similarities and differences between compound interest and continuously compounded interest.

4. **Communicate Precisely** Kylee is using a calculator to find an exponential regression model. How would you explain to Kylee what the variables in the model $y = a \cdot b^x$ represent?

Do You KNOW HOW?

The exponential function models the annual rate of increase. Find the monthly and quarterly rates.

5. $f(t) = 2{,}000(1.03)^t$

6. $f(t) = 500(1.055)^t$

Find the total amount of money in an account at the end of the given time period.

7. compounded monthly, $P = \$2{,}000$, $r = 3\%$, $t = 5$ years

8. continuously compounded, $P = \$1{,}500$, $r = 1.5\%$, $t = 6$ years

Write an exponential model given two points.

9. (3, 55) and (4, 70)

10. (7, 12) and (8, 25)

11. Paul invests \$6,450 in an account that earns continuously compounded interest at an annual rate of 2.8%. What is the value of the account after 8 years?

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UNDERSTAND

12. **Error Analysis** Suppose \$6,500 is invested in an account that earns interest at a rate of 2% compounded quarterly for 10 years. Describe and correct the error a student made when finding the value of the account.

$$A = 6500\left(1 + \frac{0.02}{12}\right)^{12(10)}$$

$$A = 7937.80$$ ✗

13. **Communicate Precisely** The points (2, 54.61) and (4, 403.48) are points on the graph of an exponential model in the form $y = a \cdot e^x$.

    a. Explain how to write the exponential model, and then write the model.

    b. How can you use the exponential model to find the value of y when $x = 8$?

14. **Model with Mathematics** Use the points listed in the table for years 7 and 8 to find an exponential model. Then use a calculator to find an exponential model for the data. Explain how to find each model. Predict the amount in the account after 15 years.

Time (yr)	Amount (\$)
1	3,225
2	3,500
3	3,754
4	4,042
5	4,368
6	4,702
7	5,063
8	5,456

15. **Higher Order Thinking** A power model is a type of function in the form $y = a \cdot x^b$. Use the points (1, 4), (2, 8), (3, 16) and (4, 64) and a calculator to find an exponential model and a power model for the data. Then use each model to predict the value of y when $x = 6$. Graph the points and models in the same window. What do you notice?

PRACTICE

Find the amount in the account for the given principal, interest rate, time, and compounding period. SEE EXAMPLES 2 AND 4

16. $P = 800$, $r = 6\%$, $t = 9$ years; compounded quarterly

17. $P = 3{,}750$, $r = 3.5\%$, $t = 20$ years; compounded monthly

18. $P = 2{,}400$, $r = 5.25\%$, $t = 12$ years; compounded semi-annually

19. $P = 1{,}500$, $r = 4.5\%$, $t = 3$ years; compounded daily

20. $P = \$1{,}000$, $r = 2.8\%$, $t = 5$ years; compounded continuously

21. $P = \$16{,}000$, $r = 4\%$, $t = 25$ years; compounded continuously

Write an exponential model given two points. SEE EXAMPLE 5

22. (9, 140) and (10, 250)

23. (6, 85) and (7, 92)

24. (10, 43) and (11, 67)

25. In 2012, the population of a small town was 3,560. The population is decreasing at a rate of 1.7% per year. How can you rewrite an exponential growth function to find the quarterly decay rate? SEE EXAMPLE 1

26. Selena took a pizza out of the oven and it started to cool to room temperature (68°F). She will serve the pizza when it reaches 150°F. She took the pizza out of the oven at 5:00 P.M. When can she serve it? SEE EXAMPLE 6

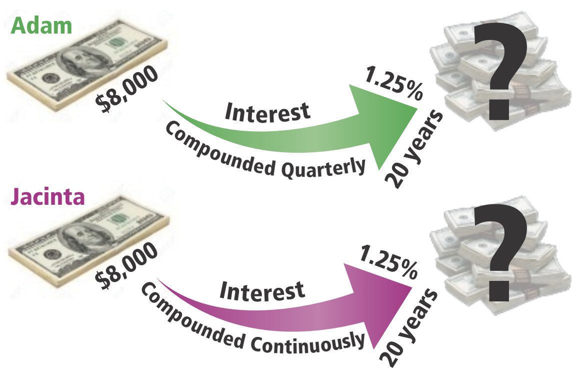
Time (min)	Temperature (°F)
5	310
8	264
10	238
15	202
20	186
25	175

PRACTICE & PROBLEM SOLVING

Practice Tutorial
Mixed Review Available Online

APPLY

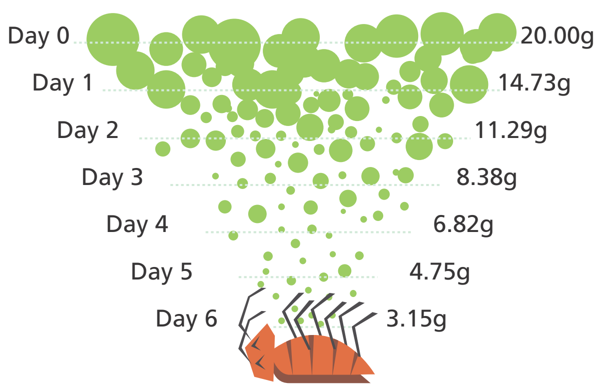
27. Reason Adam invests $8,000 in an account that earns 1.25% interest, compounded quarterly for 20 years. On the same date, Jacinta invests $8,000 in an account that earns continuous compounded interest at a rate of 1.25% for 20 years. Who do you predict will have more money in their account after 20 years? Explain your reasoning.



28. Make Sense and Persevere A blogger found that the number of visits to her Web site increases 5.6% annually. The Web site had 80,000 visits this year. Write an exponential model to represent this situation. By what percent does the number of visits increase daily? Explain how you found the daily rate.

29. Use Structure Jae invested $3,500 at a rate of 2.25% compounded continuously in 2010. How much will be in the account in 2025? How much interest will the account have earned by 2025?

30. Model with Mathematics A scientist is conducting an experiment with a pesticide. Use a calculator to find an exponential model for the data in the table. Use the model to determine how much pesticide remains after 180 days.



ASSESSMENT PRACTICE

31. The table shows the account information of five investors. Which statement is true, assuming no withdrawals are made? F-LE.2.5

Employee	P	r	t(years)	Compound
Anna	4000	1.5%	12	Quarterly
Nick	2500	3%	8	Monthly
Lori	7200	5%	15	Annually
Tara	2100	4.5%	6	Continuously
Steve	3800	3.5%	20	Semi-annually

Ⓐ After 12 years, Anna will have about $4,788.33 in her account.

Ⓑ After 8 years, Nick will have about $3,177.17 in his account.

Ⓒ After 15 years, Lori will have about $15,218.67 in her account.

Ⓓ After 20 years, Steve will have about $7,629.00 in his account.

32. SAT/ACT Rick invested money in a continuous compound account with an interest rate of 3%. How long will it take Rick's account to double?

Ⓐ about 2 years

Ⓑ about 10 years

Ⓒ about 23 years

Ⓓ about 46 years

Ⓔ about 67 years

33. Performance Task Cassie is financing a $2,400 treadmill. She is going to use her credit card for the purchase. Her card charges 17.5% interest compounded monthly. She is not required to make minimum monthly payments.

Part A How much will Cassie pay in interest if she waits a full year before paying the full balance?

Part B How much additional interest will Cassie pay if she waits two full years before paying the full balance?

Part C If both answers represent a single year of interest, why is the answer in B greater than the answer in A?