1-2 Transformations of Functions

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Activity Assess

I CAN… apply transformations to graph functions and write equations.

VOCABULARY

- compression
- reflection
- stretch
- transformation
- translation

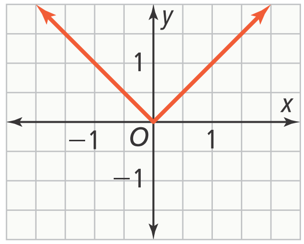
MAFS.912.F-BF.2.3–Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Also F-IF.2.5
MAFS.K12.MP.4.1, MP.5.1, MP.7.1

LOOK FOR RELATIONSHIPS

This type of transformation slides the graph up or down. You could perform a transformation like this to a nonvertical line and produce a parallel line.

EXPLORE & REASON

The graph of the function $f(x) = |x|$ is shown.



A. Graph the function $g(x) = |x + c|$ several times with different values for c (any value from −5 to 5).

B. Look for Relationships Predict what will happen to the graph if c is a number greater than 100. What if c is a number between 0 and $\frac{1}{2}$?

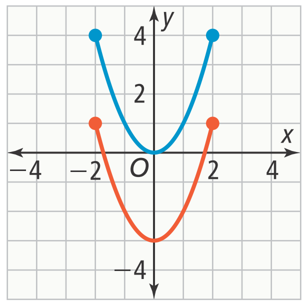
ESSENTIAL QUESTION

What do the differences between the equation of a function and the equation of its parent function tell you about the differences in the graphs of the two functions?

EXAMPLE 1 Translate a Function

A. Graph the function $f(x) = x^2$ for the domain [−2, 2]. The graph of g is the graph of f after a translation of 3 units down. How are the equations, domains, and ranges of f and g related?

Every point on the graph of g is 3 units below a corresponding point on the graph of f.



$$(x, f(x)) \rightarrow (x, f(x) - 3) = (x, g(x))$$

$$g(x) = f(x) - 3$$

The domains of f and g are the same: [−2, 2]. The range values of g are 3 units less than the range values of f. The range of f is [0, 4] and the range of g is [−3, 1].

A **translation** like this one is a particular kind of transformation of a function, one that shifts each point on a graph the same distance and direction.

Other kinds of **transformations** of a function may reflect its graph across an axis, or stretch or compress its graph.

In general, if $g(x) = f(x) + k$, then the graph of g is a vertical translation of the graph of f by k units.

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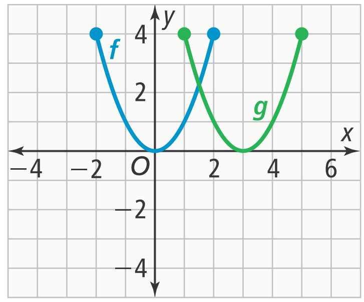
EXAMPLE 1 CONTINUED

B. Graph the function $f(x) = x^2$ for the domain [–2, 2]. The graph of the function *g* is the graph of *f* after a translation 3 units to the right. How are the equations, domains, and ranges of *f* and *g* related?



Every point on the graph of *g* is 3 units to the right of the corresponding point on the graph of *f*.

$(x, f(x)) \rightarrow (x, g(x + 3))$

The translation of the graph of *f* to the graph of *g* can be described as $g(x + 3) = f(x)$, or $g(x) = f(x - 3)$.

The range values of *f* and *g* are the same: [0, 4]. The domain values of *g* are 3 units more than the domain values of *f*. The domain of *f* is [–2, 2] and the domain of *g* is [1, 5].

In general, if $g(x) = f(x - h)$, then the graph of *g* is a horizontal translation of the graph of *f* by *h* units.

Try It! **1. a.** How did the transformation of *f* to *g* in part (a) affect the intercepts?

b. How did the transformation of *f* to *g* in part (b) affect the intercepts?

EXAMPLE 2 Reflect a Function Across the *x*- or *y*-Axis

VOCABULARY
Recall that a **reflection** is a transformation that maps each point to a new point across a given line, called the *line of reflection*. The line of reflection is the perpendicular bisector of the segment between the point and its image.

A. Graph $f(x) = 2x - 6$ and the function *g*, whose graph is the reflection of the graph of *f* across the *x*-axis. How are their equations related?

Graph *f*. Then graph *g* by reflecting each point of the graph of *f* across the *x*-axis. For each point (x, y) on the graph of *f*, plot the point $(x, -y)$ to get the graph of *g*.

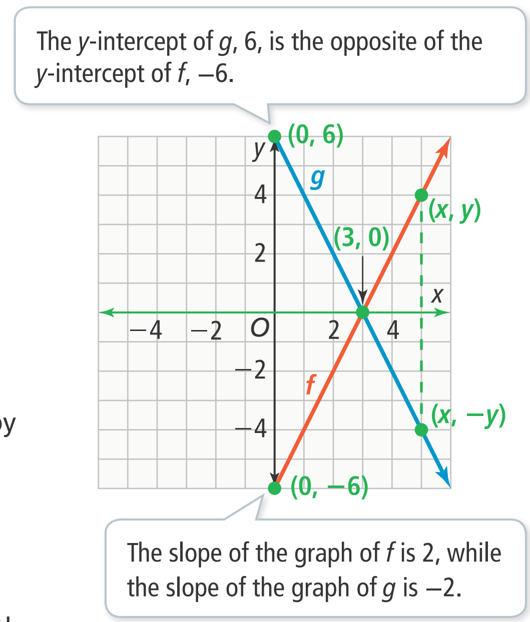
Since the y-values of the new function have the opposite sign, $g(x) = -f(x)$.

From the graph, you can see that $g(x) = -2x + 6$.

You can check that $g(x) = -f(x)$ by substituting for $f(x)$.

$$g(x) = -f(x)$$
$$= -(2x - 6)$$
$$g(x) = -2x + 6$$

The expression that defines *g* is the opposite of the expression that defines *f*.



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EXAMPLE 2 CONTINUED

B. Graph $f(x) = 2x - 6$ and the function h, whose graph is the reflection of the graph of f across the y-axis. How are their equations related?

Graph f. Then reflect every point on the graph of f over the y-axis to produce the graph of h.

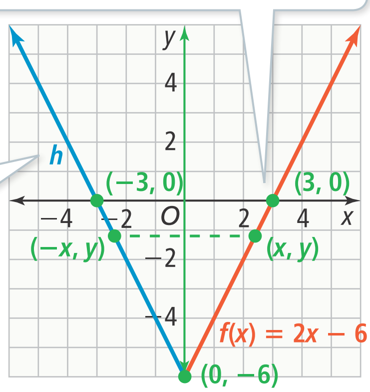
From the graph, you can see that $h(x) = -2x - 6$.

You can check that $h(x) = f(-x)$ by substituting for $f(-x)$.

$$h(x) = f(-x)$$
$$= 2(-x) - 6$$
$$h(x) = -2x - 6$$

The slope of the graph of f is 2, and the slope of the graph of h is -2.

For any point (x, y) on the graph of f, there is a reflected point $(-x, y)$ on the graph of h, so $h(x) = f(-x)$. The x-intercept of h, -3, is the opposite of the x-intercept of f, 3.



The function h has a slope that is the opposite of the slope of f but with the same y-intercept.

USE STRUCTURE
Why does $g(x) = -f(x)$ affect the y-coordinate of each point and $g(x) = f(-x)$ affect the x-coordinate of each point?

Try It! 2. What is an equation for the reflected graph? Check by graphing.

a. the graph of $f(x) = x^2 - 2$ reflected across the x-axis

b. the graph of $f(x) = x^2 - 2$ reflected across the y-axis

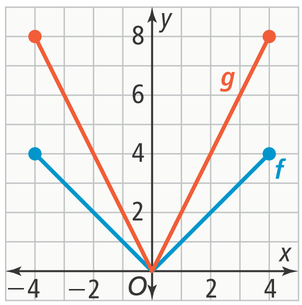
CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Understand Stretches and Compressions

A. Graph $f(x) = |x|$ with domain $[-4, 4]$ and $g(x) = 2 \bullet f(x)$. How are the domains and ranges related?

Use a table to find points on the graph of g.

x	$f(x)$	$g(x) = 2 \bullet f(x)$	$(x, g(x))$
−4	4	$2 \bullet f(-4) = 2(4) = 8$	(−4, 8)
−2	2	$2 \bullet f(-2) = 2(2) = 4$	(−2, 4)
0	0	$2 \bullet f(0) = 2(0) = 0$	(0, 0)
2	2	$2 \bullet f(2) = 2(2) = 4$	(2, 4)
4	4	$2 \bullet f(4) = 2(4) = 8$	(4, 8)



The domains of f and g are the same. Each y-value is multiplied by 2, so for the function with the given domain, the range of f, [0, 4], is doubled for g to [0, 8].

A transformation that increases the distance between the points of a graph and a given line by the same factor is called a **stretch**. The graph of g is a vertical stretch of the graph of f by a factor of 2.

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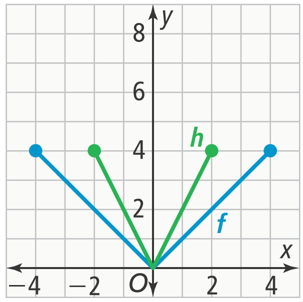
EXAMPLE 3 CONTINUED

B. Graph $f(x) = |x|$ with domain $[-4, 4]$ and $h(x) = f(2x)$. How are the domains and ranges related?

Use a table to find points on the graph of h.

x	$f(x)$	$h = f(2x)$	$(x, h(x))$
-2	2	$f(2(-2)) = f(-4) = 4$	$(-2, 4)$
-1	1	$f(2(-1)) = f(-2) = 2$	$(-1, 2)$
0	0	$f(2(0)) = f(0) = 0$	$(0, 0)$
1	1	$f(2(1)) = f(2) = 2$	$(1, 2)$
2	2	$f(2(2)) = f(4) = 4$	$(2, 4)$



COMMON ERROR

Be careful not to assume that the domain of a transformed function is the same as the domain of the original function. Notice that $h(-4)$ would equal $f(-8)$, which is outside the domain of f, $[-4, 4]$.

For each corresponding output, the value of the input for h is half the value of the input for f. The two functions have the same range, but the values in the domain of h, $[-2, 2]$, are half as large as the values in the domain of f, $[-4, 4]$.

A transformation that decreases the distance between the points of a graph and a given line by the same factor is called a **compression**. The graph of h is a horizontal compression of the graph of f by a factor of 2.

Try It! **3.** Show that $j(x) = f\left(\frac{1}{2}x\right)$ is a horizontal stretch of the graph of f.

CONCEPT Stretches and Compressions

Vertical Stretches and Compressions:

If $a > 1$, then $g(x) = a \cdot f(x)$ is a vertical stretch of f by the factor a.

If $0 < a < 1$, then $g(x) = a \cdot f(x)$ is a vertical compression of f by the factor $\frac{1}{a}$.

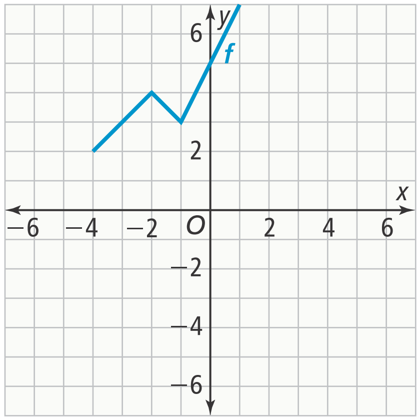
Horizontal Stretches and Compressions:

If $a > 1$, then $g(x) = f(ax)$ is a horizontal compression of f by a factor of a.

If $0 < a < 1$, then $g(x) = f(ax)$ is a horizontal stretch of f by a factor of a.

EXAMPLE 4 Graph a Combination of Transformations

The graph represents $y = f(x)$. Using $y = f(x)$, how can you graph a combination of transformations?

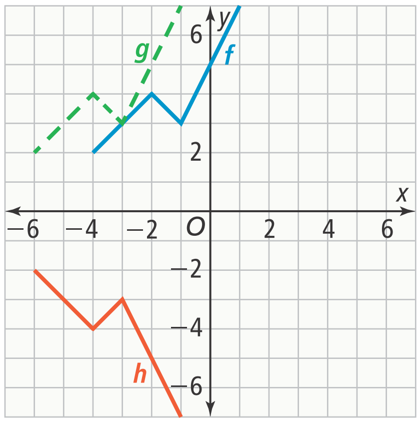


A. Graph $y = -f(x + 2)$.

Graph $g(x) = f(x + 2)$, which is a translation of f left 2 units.

Graph $h(x) = -f(x + 2)$, which is a reflection of g across the x-axis.

The graph of h is a translation of f left 2 units followed by a reflection across the x-axis.



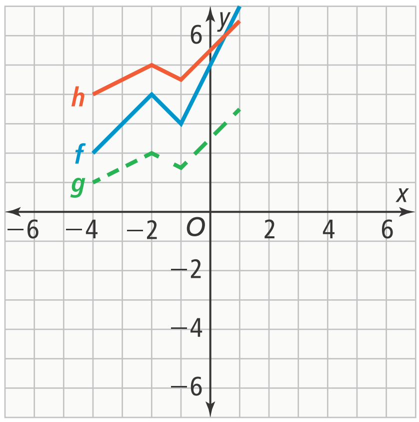
STUDY TIP
It is easier to perform one transformation at a time. Trying to perform both transformations at the same time will often result in an incorrect graph.

B. Graph $y = \frac{1}{2}f(x) + 3$.

Graph $g(x) = \frac{1}{2}f(x)$, which is a vertical compression of f by a factor of 2.

Graph $h(x) = \frac{1}{2}f(x) + 3$, which is a translation of g up 3 units.

The graph of h is a vertical compression of f by a factor of 2 followed by a translation 3 units up.



Try It! **4.** Using the graph of f above, graph each equation.

a. $y = f(2x) - 4$ **b.** $y = f(2x - 3) - 2$

EXAMPLE 5 Identify Transformations From an Equation

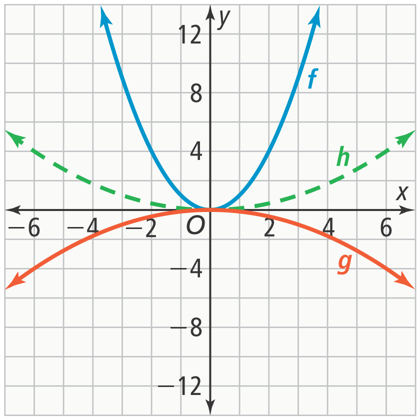
What transformations of $f(x) = x^2$ result in the graph of the function g?

A. $g(x) = -\left(\frac{1}{3}x\right)^2$

$h(x) = f\left(\frac{1}{3}x\right) = \left(\frac{1}{3}x\right)^2$ represents a horizontal stretch of the graph of f by a factor of 3.

$g(x) = -h(x) = -f\left(\frac{1}{3}x\right) = -\left(\frac{1}{3}x\right)^2$ represents a reflection across the x-axis of $f\left(\frac{1}{3}x\right)$.

The graph of g is a horizontal stretch by a factor of 3 and a reflection across the x-axis of the graph of f.

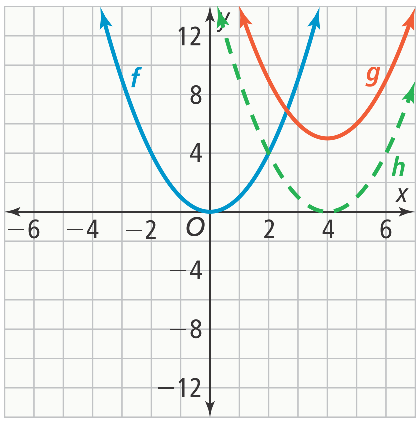


B. $g(x) = (x - 4)^2 + 5$

$h(x) = f(x - 4) = (x - 4)^2$ represents a translation 4 units to the right of the graph of f.

$g(x) = h(x) + 5 = f(x - 4) + 5 = (x - 4)^2 + 5$ represents a translation 5 units up of the graph of $f(x - 4)$.

The graph of g is a translation 4 units right and 5 units up of the graph of f.



USE APPROPRIATE TOOLS
You can use graphing technology to graph the original and transformed equations to check that the transformations you have identified are correct.

Try It! **5.** What transformations of the graph of $f(x) = |x|$ are applied to graph the function g?

a. $g(x) = \frac{1}{2}|x + 3|$ **b.** $g(x) = -|x| + 2$

APPLICATION

EXAMPLE 6 Write an Equation From a Graph

A scenic train ride makes trips on an old mining line. The graph shows the distance *y* in kilometers of the train from the station *x* minutes after the ride begins. What equation represents the distance from the station as a function of time? What is its domain?

The graph shows a reflection of an absolute value graph across the *x*-axis and a translation upward and to the right. The general form of this absolute value function is $y = -a|x - h| + k$, where the point (h, k) represents the vertex and $-a$ indicates that the graph opens downward.

Substituting the point of the vertex (15, 15) for (h, k) gives the equation $y = -a|x - 15| + 15$.

To solve for *a*, you can use any point on the graph. Using the point (0, 0) to substitute for (x, y) in the equation simplifies the computation:

$$y = -a|x - 15| + 15$$

$$0 = -a|0 - 15| + 15$$

$$0 = -15a + 15$$

$$-15 = -15a$$

$$a = 1$$

STUDY TIP
Solving algebraically is only one method for determining a stretch or compression factor.

Now you can write the equation for distance as a function of time:

$$y = -|x - 15| + 15.$$

According to the graph, the train returns to its station after 30 minutes, so the function's domain is [0, 30].

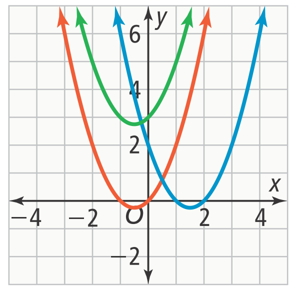
Try It! **6.** How would the graph and equation be affected if the train traveled twice as far in the same amount of time?

CONCEPT SUMMARY Transformations of Functions

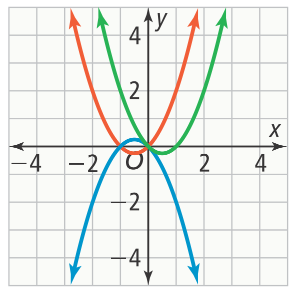
For a function $f(x)$, the graph of $f(x) = a \cdot f[b(x - h)] + k$ represents a transformation of the graph of that function by translation, reflection, or stretching.

WORDS	EQUATIONS
Horizontal translation of *f* right 2 units (altering *h*)	$f(x)$ becomes $g(x) = f(x - 2)$ $f(x) = x^2 + x$ $g(x) = (x - 2)^2 + (x - 2) = x^2 - 3x + 2$
Vertical translation of *f* up 3 units (altering *k*)	$f(x)$ becomes $h(x) = f(x) + 3$ $f(x) = x^2 + x$ $h(x) = x^2 + x + 3$

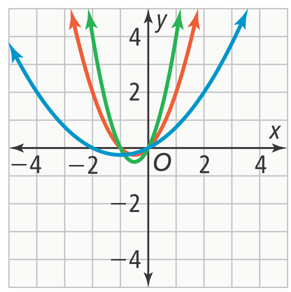
GRAPHS



WORDS	EQUATIONS
Reflection of *f* across the *x*-axis (altering *a*)	$f(x)$ becomes $-f(x)$ $f(x) = x^2 + x$ $-f(x) = -(x^2 + x) = -x^2 - x$
Reflection of *f* across the *y*-axis (altering *b*)	$f(x)$ becomes $f(-x)$ $f(x) = x^2 + x$ $f(-x) = (-x)^2 + (-x) = x^2 - x$



WORDS	EQUATIONS
Horizontal stretch of *f* by a factor of 2 (altering *b*)	$f(x)$ becomes $f\left(\frac{1}{2}x\right)$ $f(x) = x^2 + x$ $f\left(\frac{1}{2}x\right) = \left(\frac{1}{2}x\right)^2 + \frac{1}{2}x = \frac{1}{4}x^2 + \frac{1}{2}x$
Vertical stretch of *f* by a factor of 2 (altering *a*)	$f(x)$ becomes $2f(x)$ $f(x) = x^2 + x$ $2f(x) = 2(x^2 + x) = 2x^2 + 2x$



Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What do the differences between the equation of a function and the equation of its parent function tell you about the differences in the graphs of the two functions?

2. **Reason** Do k and h affect the input or output for $g(x) = f(x) + k$ and $g(x) = f(x - h)$? Explain.

3. **Error Analysis** Margo is comparing the functions $f(x) = |x|$ and $g(x) = |x + 1| - 5$. She said the graph of g is a vertical translation of the graph of f 5 units down and a horizontal translation of the graph of f 1 unit right. What is Margo's error?

Do You KNOW HOW?

Graph each function and its parent function.

4. $g(x) = |x| - 1$
5. $g(x) = (x - 3)^2$
6. $g(x) = -|x|$
7. $g(x) = -x$
8. $g(x) = x^2 - 2$
9. $g(x) = \frac{1}{2}|x|$
10. $g(x) = 4x$
11. $g(x) = |5x|$
12. Which types of transformations in Exercises 4–11 do not change the shape of a graph? Which types of transformations change the shape of a graph? Explain.

PRACTICE & PROBLEM SOLVING

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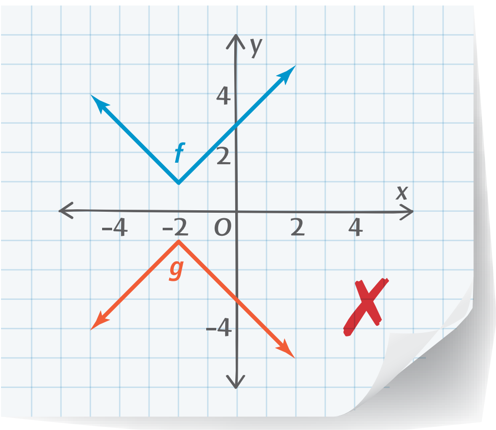
Practice Tutorial

Additional Exercises Available Online

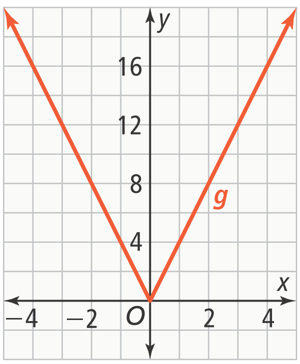
UNDERSTAND

13. **Use Structure** Write a function g with the parent function $f(x) = x^2$ that has a vertex at $(3, -6)$.

14. **Error Analysis** Describe and correct the error a student made in graphing $g(x) = f(-x)$ as a reflection across the y-axis of the graph of $f(x) = |x + 2| + 1$.



15. **Higher Order Thinking** Describe the transformation g of $f(x) = |x|$ as a stretch and as a compression. Then write two equations to represent the function. What can you conclude? Explain.



16. **Use Structure** The graph of the parent function $f(x) = x^2$ is reflected across the y-axis. Write an equation for the function g after the reflection. Show your work. Based on your equation, what happens to the graph? Explain.

17. **Error Analysis** Monisha is comparing $f(x) = |x|$ and $g(x) = |2x - 4|$. She said the graph of g is a horizontal translation of the graph of f 4 units to the right and a horizontal compression of the graph of f by a factor of 2. What is Monisha's error?

PRACTICE

Graph each function as a translation of its parent function, f. How did the transformation affect the domain and range? SEE EXAMPLE 1

18. $g(x) = |x| - 5$

19. $g(x) = (x + 1)^2$

20. $g(x) = |x - 3|$

21. $g(x) = x^2 + 2$

What is the equation for the image graph? Check by graphing. SEE EXAMPLE 2

22. Reflect $f(x) = x^2 + 1$ across the x-axis.

23. Reflect $f(x) = x^2 + 1$ across the y-axis.

Graph each function as a vertical stretch or compression of its parent function. SEE EXAMPLE 3

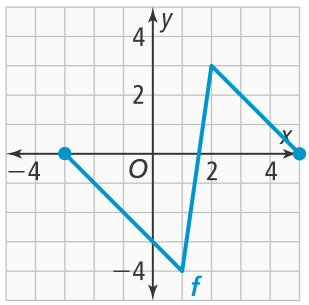
24. $g(x) = 0.25|x|$

25. $g(x) = 3x^2$

26. $g(x) = 1.5|x|$

27. $g(x) = 0.75x^2$

28. Use the graph of $f(x)$ to graph $y = f(x + 1) + 2$. SEE EXAMPLE 4



What transformations of $f(x) = x^2$ are applied to the function g? SEE EXAMPLE 5

29. $g(x) = 2(x + 1)^2$

30. $g(x) = (x - 3)^2 + 5$

31. $g(x) = -x^2 - 6$

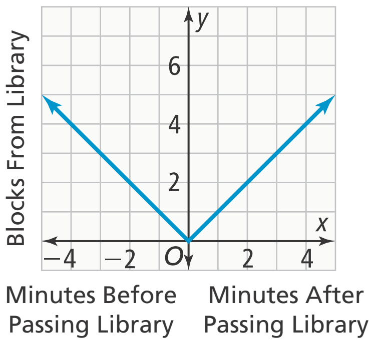
32. $g(x) = 4(x - 7)^2 - 9$

33. The graph shows the height y in feet of a flying insect x seconds after taking off from the ground. Write an equation that represents the height of the insect as a function of time. SEE EXAMPLE 6

PRACTICE & PROBLEM SOLVING

APPLY

34. Model With Mathematics Chiang walks to school each day. She passes the library halfway on her walk to school. She walks at a rate of 1 block per minute. The graph shows the distance Chiang is from the library as she walks to school.

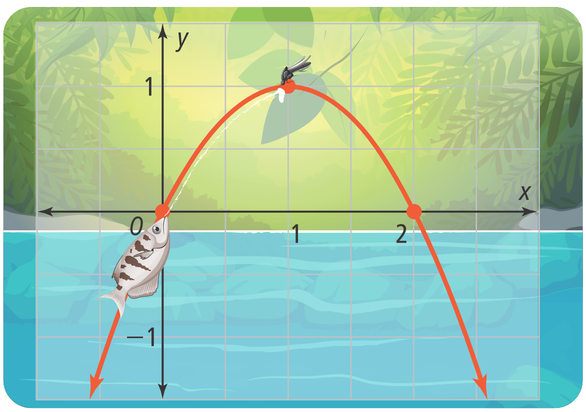


a. Write a function, f, to model the distance Chiang is from the library when she walks to school.

b. If Chiang jogs to school, she travels at a rate of 2.5 blocks per minute. Write a function, g, to model the distance Chiang is from the library when she jogs to school.

c. Graph the function, g, that models the distance Chiang is from the library when she jogs to school.

35. Model With Mathematics The archer fish spits water at flying insects to knock them into the water. The path of the water is shown with x and y distances in feet. Write an equation to represent the path of the water in relation to the coordinate grid. Then determine the coordinates of the point of maximum height of the water.



ASSESSMENT PRACTICE

36. The first table shows values of x and $f(x)$ for a given function f. Copy and complete the second table. F-BF.2.3

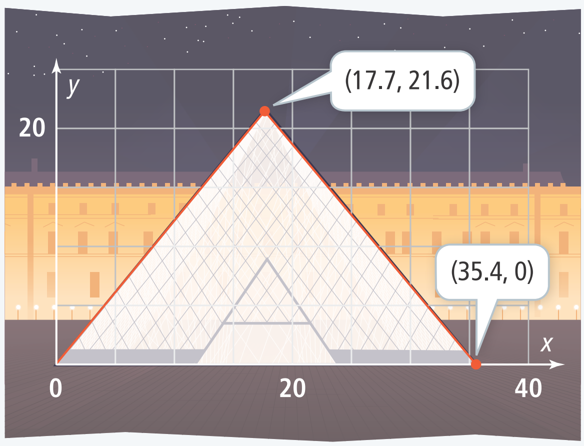
x	$f(x)$
−2	4
−1	1
0	0
1	1
2	4

x	$f(2x) + 3$
−2	
−1	
0	
1	
2	

37. SAT/ACT Which translation is part of transforming $f(x) = x^2$ into $h(x) = (x + 4)^2 - 2$?

Ⓐ left 4 units
Ⓑ left 2 units
Ⓒ right 2 units
Ⓓ right 4 units

38. Performance Task The Louvre Pyramid in Paris is shown on the coordinate grid, where x and y are measured in meters and the ground is represented by the x-axis.



Part A The outline of the Pyramid is a transformation of the function $f(x) = |x|$. Write a function g to model the outline of the Pyramid.

Part B What is the domain and range of the function that models the outline of the Pyramid? What do the domain and range represent?