3-3 Polynomial Identities

PearsonRealize.com

I CAN… prove and use polynomial identities.

VOCABULARY

- Binomial Theorem
- identity
- Pascal's Triangle

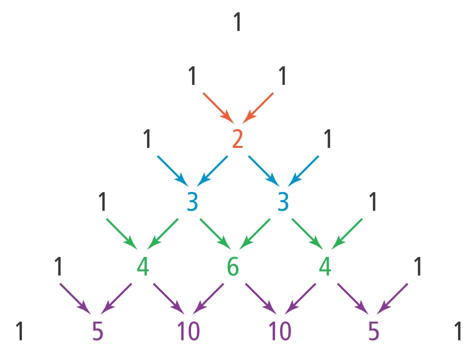
MAFS.912.A-APR.3.4–Prove polynomial identities and use them to describe numerical relationships.
Also A-SSE.1.2, A-APR.3.5
MAFS.K12.MP.2.1, MP.3.1, MP.7.1

Activity Assess

EXPLORE & REASON

Look at the following triangle.

Each number is the sum of the two numbers diagonally above. If there is not a second number, think of it as 0.



A. Write the numbers in the next three rows.

B. **Look for Relationships** What other patterns do you see?

C. Find the sum of the numbers in each row of the triangle. Write a formula for the sum of the numbers in the n^{th} row.

ESSENTIAL QUESTION

How can you use polynomial identities to rewrite expressions efficiently?

CONCEPT Polynomial Identities

A mathematical statement that equates two polynomial expressions is an **identity** if one side can be transformed into the other side using mathematical operations. These polynomial identities are helpful tools used to multiply and factor polynomials.

Difference of Squares

$a^2 - b^2 = (a + b)(a - b)$

Example: $25x^2 - 36y^2$

Substitute $5x$ for a and $6y$ for b.

$25x^2 - 36y^2 = (5x + 6y)(5x - 6y)$

Square of a Sum

$(a + b)^2 = a^2 + 2ab + b^2$

Example: $(3x + 4y)^2$

Substitute $3x$ for a and $4y$ for b.

$(3x + 4y)^2 = (3x)^2 + 2(3x)(4y) + (4y)^2$

$= 9x^2 + 24xy + 16y^2$

Difference of Cubes

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Example: $8m^3 - 27$

Substitute $2m$ for a and 3 for b.

$8m^3 - 27 = (2m - 3)[(2m)^2 + (2m)(3) + 3^2]$

$= (2m - 3)(4m^2 + 6m + 9)$

Sum of Cubes

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Example: $g^3 + 64h^3$

Substitute g for a and $4h$ for b.

$g^3 + 64h^3 = (g + 4h)[g^2 - (g)(4h) + (4h)^2]$

$= (g + 4h)(g^2 - 4gh + 16h^2)$

EXAMPLE 1 Prove a Polynomial Identity

How can you prove the Sum of Cubes Identity, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$?

To prove an identity, start with the expression on one side of the equation and use properties of operations on polynomials to transform it into the expression on the other side.

$(a + b)(a^2 - ab + b^2)$

$= a(a^2 - ab + b^2) + b(a^2 - ab + b^2)$ Use the Distributive Property.

$= a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$ Use the Distributive Property.

$= a^3 + (-a^2b + a^2b) + (ab^2 - ab^2) + b^3$ Group like terms.

$= a^3 + b^3$ Combine like terms.

So, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

USE STRUCTURE
Another way to establish the identity is to multiply each term of the second factor by $(a + b)$, and then combine like terms.

Try It! **1.** Prove the Difference of Cubes Identity.

EXAMPLE 2 Use Polynomial Identities to Multiply

How can you use polynomial identities to multiply expressions?

A. $(2x^2 + y^3)^2$

The sum is a binomial, and the entire sum is being raised to the second power.

Use the Square of a Sum Identity to find the product:

$(a + b)^2 = a^2 + 2ab + b^2$

$(2x^2 + y^3)^2 = (2x^2)^2 + 2(2x^2)(y^3) + (y^3)^2$ Substitute $2x^2$ for a and y^3 for b.

$= 4x^4 + 4x^2y^3 + y^6$ Simplify.

So, $(2x^2 + y^3)^2 = 4x^4 + 4x^2y^3 + y^6$.

COMMON ERROR
When finding $(a + b)^2$, recall that it is not sufficient to square the first term and square the second term. You must distribute the two binomials.

B. $41 \bullet 39$

Rewrite the expression in terms of a and b.

$41 \bullet 39 = (a + b)(a - b)$

$= (40 + 1)(40 - 1)$

Use the Difference of Squares Identity:

$(40 + 1)(40 - 1) = 40^2 - 1^2$

$= 1{,}600 - 1$

$= 1{,}599$

So $41 \bullet 39 = 1{,}599$.

Try It! **2.** Use polynomial identities to multiply the expressions.

a. $(3x^2 + 5y^3)(3x^2 - 5y^3)$ **b.** $(12 + 15)^2$

EXAMPLE 3 Use Polynomial Identities to Factor and Simplify

How can you use polynomial identities to factor polynomials and simplify numerical expressions?

A. $9m^4 - 25n^6$

$9m^4$ and $25n^6$ are both perfect squares.

A square term includes an even exponent, not necessarily an exponent that is a perfect square.

$9m^4 = (3m^2)^2$

$25n^6 = (5n^3)^2$

Use the Difference of Squares Identity: $a^2 - b^2 = (a + b)(a - b)$.

$9m^4 - 25n^6 = (3m^2)^2 - (5n^3)^2$ Express each term as a square.

$= (3m^2 + 5n^3)(3m^2 - 5n^3)$ Write the factors.

So, $9m^4 - 25n^6 = (3m^2 + 5n^3)(3m^2 - 5n^3)$.

B. $x^3 - 216$

x^3 and 216 are both perfect cubes.

$x^3 = (x)^3$

$216 = 6^3$

Use the Difference of Cubes Identity: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

$x^3 - 216 = (x)^3 - (6)^3$ Express each term as a cube.

$= (x - 6)(x^2 + 6x + 36)$ Write the factors.

So, $x^3 - 216 = (x - 6)(x^2 + 6x + 36)$.

COMMON ERROR
The second factor is *almost* a Square of a Sum. Remember that the middle term of the Difference of Cubes Identity is the product *ab*, not 2*ab*.

C. $11^3 + 5^3$

Use the Sum of Cubes Identity: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

$11^3 + 5^3 = (11 + 5)(11^2 - 11(5) + 5^2)$

$= (16)(121 - 55 + 25)$

$= 16(91)$

$= 1,456$

So, $11^3 + 5^3 = 1,456$.

Try It! 3. Use polynomial identities to factor each polynomial.

a. $m^8 - 9n^{10}$ **b.** $27x^9 - 343y^6$ **c.** $12^3 + 2^3$

Activity Assess

CONCEPTUAL UNDERSTANDING

EXAMPLE 4 Expand a Power of a Binomial

How is $(x + y)^n$ obtained from $(x + y)^{n-1}$?

A. What are $(x + y)^3$ and $(x + y)^4$?

$$(x + y)^3 = (x + y)(x + y)^2$$
$$= (x + y)(x^2 + 2xy + y^2)$$

$$1x^2 \quad + \quad 2xy \quad + \quad 1y^2$$
$$(x \; + \; y) \quad (x \; + \; y) \quad (x \; + \; y)$$
$$x^3 + x^2y + 2x^2y + 2xy^2 + xy^2 + y^3$$

Multiply each term by $(x + y)$ to get $(x + y)^3$.

$$1x^3 + 3x^2y + 3xy^2 + 1y^3 = (x + y)^3$$
$$(x \quad y) \quad (x \quad y) \quad (x \quad y) \quad (x \quad y)$$
$$x^4 + x^3y + 3x^3y + 3x^2y^2 + 3x^2y^2 + 3xy^3 + xy^3 + y^4$$

Multiply each term in $(1x^3 + 3x^2y + 3xy^2 + y^3)$ by $(x + y)$ to get $(x + y)^4$.

$$1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 = (x + y)^4$$

The coefficients of $(x + y)^n$ are produced by adding the coefficients of $(x + y)^{n-1}$, producing an array known as Pascal's Triangle. **Pascal's Triangle** is the triangular pattern of numbers where each number is the sum of the two numbers diagonally above it. If there is not a second number diagonally above in the triangle, think of the missing number as 0.

Row 0	1	1	$(x + y)^0$
Row 1	1 1	$1x + 1y$	$(x + y)^1$
Row 2	1 2 1	$1x^2 + 2xy + 1y^2$	$(x + y)^2$
Row 3	1 3 3 1	$1x^3 + 3x^2y + 3xy^2 + 1y^3$	$(x + y)^3$
Row 4	1 4 6 4 1	$1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$	$(x + y)^4$

You can obtain $(x + y)^n$ by adding adjacent pairs of coefficients from $(x + y)^{n-1}$.

STUDY TIP

Notice the patterns of the powers. The powers of x decrease from n to 0 and the powers of y increase from 0 to n when reading the terms from left to right.

B. Use Pascal's Triangle to expand $(x + y)^5$.

Add pairs of coefficients from Row 4 to complete Row 5.

Row 4	1	4	6	4	1	
Row 5	1	5	10	10	5	1

Write the expansion. Use the coefficients from Row 5 with powers of x starting at 5 and decreasing to 0 and with powers of y starting at 0 and increasing to 5.

The sum of the exponents in each term is equal to the exponent on the original binomial.

$$(x + y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$

Try It! **4.** Use Pascal's Triangle to expand $(x + y)^6$.

CONCEPT Binomial Theorem

The **Binomial Theorem** states that, for every positive integer n,

$$(a + b)^n = C_0a^n + C_1a^{n-1}b + C_2a^{n-2}b^2 + \ldots + C_{n-1}ab^{n-1} + C_nb^n.$$

The coefficients $C_0, C_1, C_2, \ldots, C_{n-1}, C_n$ are the numbers in Row n of Pascal's Triangle.

Notice that the powers of a are decreasing while the powers of b are increasing, and that the sum of the powers of a and b in each term is always n.

EXAMPLE 5 Apply the Binomial Theorem

Use the Binomial Theorem to expand the expressions.

A. Find $(x - 3)^4$.

Step 1 Use the Binomial Theorem to write the expansion when $n = 4$.

$$C_0a^4 + C_1a^3b + C_2a^2b^2 + C_3ab^3 + C_4b^4$$

Step 2 Use Row 4 in Pascal's Triangle to write the coefficients.

$$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Step 3 Identify a and b.

$$a = x \text{ and } b = -3$$

Step 4 Substitute x for a and -3 for b in the pattern. Then simplify.

$$x^4 + 4x^3(-3) + 6x^2(-3)^2 + 4x(-3)^3 + (-3)^4$$

$$x^4 - 12x^3 + 54x^2 - 108x + 81$$

So $(x - 3)^4 = x^4 - 12x^3 + 54x^2 - 108x + 81$.

COMMON ERROR
Remember that the base of $(a + b)^n$ in the Binomial Theorem is $(a + b)$. If the terms are being subtracted, use the opposite of b in the expansion.

B. Find $(s^2 + 3)^5$.

The expansion of $(a + b)^5$ is $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

Since $a = s^2$ and $b = 3$, the expansion is:

$$(s^2 + 3)^5 = (s^2)^5 + 5(s^2)^4(3) + 10(s^2)^3(3)^2 + 10(s^2)^2(3)^3 + 5(s^2)(3)^4 + (3)^5$$

$$= s^{10} + 15s^8 + 90s^6 + 270s^4 + 405s^2 + 243$$

So $(s^2 + 3)^5 = s^{10} + 15s^8 + 90s^6 + 270s^4 + 405s^2 + 243$.

Try It! **5.** Use the Binomial Theorem to expand each expression.

a. $(x - 1)^7$

b. $(2c + d)^6$

CONCEPT SUMMARY Polynomial Identities

POLYNOMIAL IDENTITIES

Special polynomial identities can be used to multiply and factor polynomials.

Difference of Squares

$a^2 - b^2 = (a + b)(a - b)$

Square of a Sum

$(a + b)^2 = a^2 + 2ab + b^2$

Difference of Cubes

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Sum of Cubes

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

BINOMIAL EXPANSION

The binomial expansion of $(a + b)^n$ has the following properties:

1) The expansion contains $n + 1$ terms.

2) The coefficients of each term are numbers from the nth row of Pascal's Triangle.

3) The exponent of a is n in the first term and decreases by 1 in each successive term.

4) The exponent of b is 0 in the first term and increases by 1 in each successive term.

5) The sum of the exponents in any term is n.

Row 0	1	1	$(x + y)^0$
Row 1	1 1	$1x + 1y$	$(x + y)^1$
Row 2	1 2 1	$1x^2 + 2xy + 1y^2$	$(x + y)^2$
Row 3	1 3 3 1	$1x^3 + 3x^2y + 3xy^2 + 1y^3$	$(x + y)^3$
Row 4	1 4 6 4 1	$1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$	$(x + y)^4$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you use polynomial identities to rewrite expressions efficiently?

2. **Reason** Explain why the middle term of $(x + 5)^2$ is $10x$.

3. **Communicate Precisely** How are Pascal's Triangle and a binomial expansion, such as $(a + b)^5$, related?

4. **Use Structure** Explain how to use a polynomial identity to factor $8x^6 - 27y^3$.

5. **Make Sense and Persevere** What number does C_3 represent in the expansion $C_0a^5 + C_1a^4b + C_2a^3b^2 + C_3a^2b^3 + C_4ab^4 + C_5b^5$? Explain.

6. **Error Analysis** Dakota said the third term of the expansion of $(2g + 3h)^4$ is $36g^2h^2$. Explain Dakota's error. Then correct the error.

Do You KNOW HOW?

Use polynomial identities to multiply each expression.

7. $(2x + 8y)(2x - 8y)$

8. $(x + 3y^3)^2$

Use polynomial identities to factor each polynomial.

9. $36a^6 - 4b^2$

10. $8x^6 - y^3$

11. $m^9 + 27n^6$

Find the term of the binomial expansion.

12. fifth term of $(x + y)^5$

13. third term of $(a - 3)^6$

Use Pascal's Triangle to expand each expression.

14. $(x + 1)^5$

15. $(a - b)^6$

Use the Binomial Theorem to expand each expression.

16. $(d - 1)^4$

17. $(x + y)^7$

UNDERSTAND

18. **Use Structure** Expand $(3x + 4y)^3$ using Pascal's Triangle and the Binomial Theorem.

19. **Error Analysis** Emma factored $625g^{16} - 25h^4$. Describe and correct the error Emma made in factoring the polynomial.

$625g^{16} - 25h^4$

$= (25g^4)^2 - (5h^2)^2$

$= (25g^4 + 5h^2)(25g^4 - 5h^2)$

✗

20. **Higher Order Thinking** Use Pascal's Triangle and the Binomial Theorem to expand $(x + i)^4$. Justify your work.

21. **Use Structure** Expand the expression $(2x - 1)^4$. What is the sum of the coefficients?

22. **Error Analysis** A student says that the expansion of the expression $(-4y + z)^7$ has seven terms. Describe and correct the error the student may have made.

23. **Reason** The sum of the coefficients in the expansion of the expression $(a + b)^n$ is 64. Use Pascal's Triangle to find the value of n.

24. **Use Structure** Factor $x^3 - 125y^6$ in the form $(x - A)(x^2 + Bx + C)$. What are the values of A, B, and C?

25. **Generalize** How many terms will there be in the expansion of the expression $(x + 3)^n$? Explain how you know.

26. **Make Sense and Persevere** How could you use polynomial identities to factor the expression $x^6 - y^6$?

PRACTICE

27. Prove the polynomial identity.
$x^4 - y^4 = (x - y)(x + y)(x^2 + y^2)$

SEE EXAMPLE 1

Use polynomial identities to multiply the expressions. SEE EXAMPLE 2

28. $(x + 9)(x - 9)$

29. $(x + 6)^2$

30. $(3x - 7)^2$

31. $(2x - 5)(2x + 5)$

32. $(4x^2 + 6y^2)(4x^2 - 6y^2)$

33. $(x^2 + y^6)^2$

34. $(8 - x^2)(8 + x^2)$

35. $(6 - y^3)^2$

36. $18 \bullet 22$

37. $103 \bullet 97$

38. $(7 + 9)^2$

39. $(10 + 5)^2$

Use polynomial identities to factor the polynomials or simplify the expressions. SEE EXAMPLE 3

40. $x^8 - 9$

41. $x^9 - 8$

42. $8x^3 + y^9$

43. $x^6 - 27y^3$

44. $4x^2 - y^6$

45. $216 + 27y^{12}$

46. $64x^3 - 125y^6$

47. $\frac{1}{16}x^6 - 25y^4$

48. $9^3 + 6^3$

49. $10^3 + 5^3$

50. $10^3 - 3^3$

51. $8^3 - 2^3$

Use the Binomial Theorem to expand the expressions. SEE EXAMPLES 4 and 5

52. $(x + 3)^3$

53. $(2a - b)^5$

54. $\left(b - \frac{1}{2}\right)^4$

55. $(x^2 + 1)^4$

56. $\left(2x + \frac{1}{3}\right)^3$

57. $(x^3 + y^2)^6$

58. $(d - 3)^4$

59. $(2m + 2n)^6$

60. $(n + 5)^5$

61. $(3x - 0.2)^3$

62. $(4g + 2h)^4$

63. $\left(m^2 + \frac{1}{2}n\right)^3$

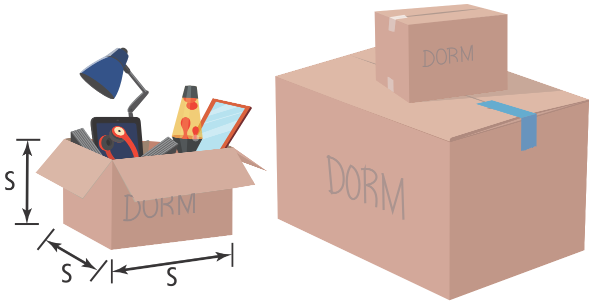
Practice Tutorial

Mixed Review Available Online

PRACTICE & PROBLEM SOLVING

APPLY

64. Reason A medium-sized shipping box with side length s units has a volume of s^3 cubic units.



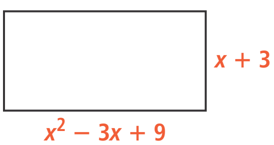
a. A large shipping box has side lengths that are 3 units longer than the medium shipping box. Write a binomial expression for the volume of the large shipping box.

b. Expand the polynomial in part a to simplify the volume of the large shipping box.

c. A small shipping box has side lengths that are 2 units shorter than the medium shipping box. Write a binomial expression for the volume of the small shipping box.

d. Expand the polynomial in part c to simplify the volume of the small shipping box.

65. Use Structure The dimensions of a rectangle are shown. Write the area of the rectangle as a sum of cubes.



66. A Pythagorean triple is a set of three positive integers a, b, and c that satisfy $a^2 + b^2 = c^2$. The identity $(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2$ can be used to generate Pythagorean triples. Use the identity to generate a Pythagorean triple when $x = 5$ and $y = 4$.

ASSESSMENT PRACTICE

67. Select all the perfect-square trinomials.
A-APR.3.4

☐ **A.** $x^2 + 16x + 64$

☐ **B.** $4x^2 - 44x + 121$

☐ **C.** $9x^2 - 15x + 25$

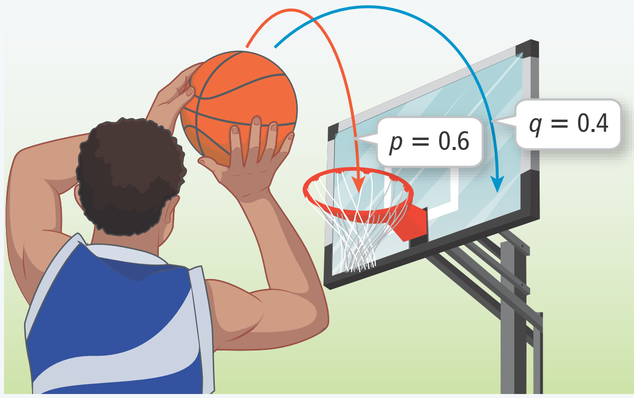
☐ **D.** $4x^2 + 64x + 16$

☐ **E.** $9x^2 - 42x + 49$

68. SAT/ACT How many terms are in the expansion of $(2x + 7y)^9$?

Ⓐ 2 Ⓑ 7 Ⓒ 8 Ⓓ 9 Ⓔ 10

69. Performance Task If an event has a probability of success p and a probability of failure q, then each term in the expansion of $(p + q)^n$ represents a probability. For example, if a basketball player makes 60% of his free throw attempts, $p = 0.6$ and $q = 0.4$. To find the probability the basketball player will make exactly h out of k free throws, find $C_{k-h}p^hq^{k-h}$, where C_{k-h} is a coefficient of row k of Pascal's Triangle, p is the probability of success, and q is the probability of failure.



Part A What is the probability the basketball player will make exactly 6 out of 10 free throws? Round to the nearest percent.

Part B Another basketball player makes 80% of her free throw attempts. Write an expression to find the probability of this basketball player making exactly 7 out of 10 free throws. Describe what each variable in the expression represents.

Part C Find the probability that the basketball player from Part B will make exactly 7 out of 10 free throws. Round to the nearest percent.