Activity Assess

2-4 Complex Numbers and Operations

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I CAN… solve problems with complex numbers.

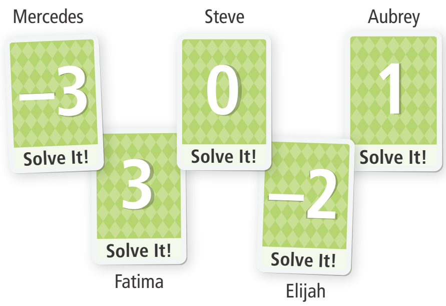
VOCABULARY

- complex conjugates
- complex number
- imaginary number
- imaginary unit *i*

MAFS.912.N-CN.1.1–Know there is a complex number *i* such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real. Also N-CN.1.2
MAFS.K12.MP.4.1, MP.7.1, MP.8.1

EXPLORE & REASON

A math class played a game called "Solve It, You're Out." At the start of each round, students chose a card from a deck marked with integers from –5 to 5. When an equation is shown, any student whose card states the solution to the equation is eliminated. Five students remain.



A. The next equation presented was $x^2 = 9$. Which student(s) was eliminated? Explain.

B. Construct Arguments In the next round, the equation presented was $x^2 = -4$. Elijah thought he was eliminated, but this is not the case. Explain why Elijah was incorrect.

C. What is true about solutions to $x^2 = a$ when a is a positive number? When a is a negative number? What about when $a = 0$?

ESSENTIAL QUESTION

How can you represent and operate on numbers that are not on the real number line?

EXAMPLE 1 Solve a Quadratic Equation Using Square Roots

How can you use square roots to solve each equation?

A. $x^2 = 16$

Notice that each side of the equation involves a perfect square.

$$x^2 = 16$$
$$x = \pm\sqrt{16}$$
$$= \pm 4$$

What numbers can you square that result in 16?

The solutions of the equation $x^2 = 16$ are 4 and –4.

B. $x^2 = -9$

There are no real numbers that you can square that result in –9. However, you can simplify the expression by extending the properties of radicals.

$$x^2 = -9$$
$$x = \pm\sqrt{-9}$$
$$x = \pm\sqrt{9}\sqrt{-1}$$
$$x = \pm 3\sqrt{-1}$$

STUDY TIP
You can also solve this equation by subtracting 16 from both sides, then factoring the expression.

The solutions of the equation $x^2 = -9$ are not real numbers but are part of a number system called the complex numbers. The number $\sqrt{-1}$ is called the **imaginary unit *i***. Replacing $\sqrt{-1}$ with i allows you to write the solutions to the equation $x^2 = -9$ as $3i$ and $-3i$.

Try It!

1. Use square roots to solve each equation. Write your solutions using the imaginary unit, *i*.

a. $x^2 = -5$

b. $x^2 = -72$

CONCEPT Complex Numbers

The imaginary unit, i, is the principal square root of -1. Then $i^2 = -1$.

An **imaginary number** is any number, bi, where b is a non-zero real number and i is the square root of -1.

Complex numbers are numbers that can be written in the form $a + bi$, where a and b are real numbers and i is the square root of -1. They include all real and imaginary numbers, as well as the sum of real and imaginary numbers.

For example:

$-6 + 4i$ $\quad (a = -6, b = 4)$

$7 - i\sqrt{2}$ $\quad (a = 7, b = -\sqrt{2})$

$0.5i$ $\quad (a = 0, b = 0.5)$

EXAMPLE 2 Add and Subtract Complex Numbers

How can you add and subtract complex numbers?

A. What is the sum of $(4 - 7i)$ and $(-11 + 9i)$?

When adding (or subtracting) two numbers in the form $a + bi$, combine the real parts and then combine the imaginary parts. The sum (or difference) may include both a real and imaginary part and can be written in the form $a + bi$.

STUDY TIP
Combine real parts and imaginary parts of complex numbers as you would combine like terms.

$$(4 - 7i) + (-11 + 9i) = (4 + -11) + (-7i + 9i)$$
$$= -7 + 2i$$

B. What is the difference of $(6 + 8i)$ and $(2 - 5i)$?

$$(6 + 8i) - (2 - 5i) = (6 + 8i) + (-2 + 5i)$$
$$= (6 + -2) + (8i + 5i)$$
$$= 4 + 13i$$

Remember to distribute the negative over the complex number.

Try It! **2.** Find the sum or difference.

a. $(-4 + 6i) + (-2 - 9i)$

b. $(3 - 2i) - (-4 + i)$

Activity Assess

EXAMPLE 3 Multiply Complex Numbers

How can you write each product in the form $a + bi$?

A. $-2.5i(8 - 9i)$

$-2.5i(8 - 9i) = -2.5i(8) - 2.5i(-9i)$ .... Use the Distributive Property.

$= -20i + 22.5i^2$ .... Multiply.

$= -20i + 22.5(-1)$ .... Simplify using the definition of i^2.

$= -22.5 - 20i$ .... Write in the form $a + bi$.

The product is $-22.5 - 20i$.

COMMON ERROR
Recall that $i^2 = -1$, so the product of 22.5 and i^2 is -22.5, not 22.5.

B. $(3 - 2i)(3 + 2i)$

$(3 - 2i)(3 + 2i) = 3(3 + 2i) - 2i(3 + 2i)$ .... Use the Distributive Property.

$= 9 + 6i - 6i - 4i^2$ .... Use the Distributive Property.

$= 9 + 6i - 6i - 4(-1)$ .... Simplify using the definition of i^2.

$= 13$ .... Simplify.

The product is 13.

Try It! 3. Write each product in the form $a + bi$.

a. $\frac{2}{5}i\left(10 - \frac{5}{2}i\right)$

b. $\left(\frac{1}{2} + 2i\right)\left(\frac{1}{2} - 2i\right)$

CONCEPT Complex Conjugates

Complex conjugates are complex numbers with equivalent real parts and opposite imaginary parts. Their product is a real number.

For example:

$7 - 8i, 7 + 8i$ $\quad$ $-2 + i, -2 - i$

$(a + bi)(a - bi)$

$a^2 - abi + abi - b^2i^2$

$a^2 - b^2(-1)$

$a^2 + b^2$

EXAMPLE 4 Simplify a Quotient With Complex Numbers

How can you write the quotient $\frac{10}{2 - i}$ in the form $a + bi$?

When the denominator has an imaginary component, you can create an equivalent fraction with a real denominator by multiplying by its complex conjugate.

STUDY TIP
Multiplying the denominator by its complex conjugate will result in a new denominator that is a real number.

$\frac{10}{2 - i} = \frac{10}{2 - i} \times \frac{2 + i}{2 + i}$ .... Use the complex conjugate of the denominator to multiply by 1.

$= \frac{10(2 + i)}{4 + 2i - 2i - i^2}$ .... Use the Distributive Property.

$= \frac{10(2 + i)}{4 + 2i - 2i - (-1)}$ .... Simplify using the definition of i^2.

$= \frac{10(2 + i)}{5}$ .... Simplify.

$= 2(2 + i)$ .... Simplify.

$= 4 + 2i$ .... Write in the form $a + bi$.

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Try It! **4.** Write each quotient in the form $a + bi$.

a. $\frac{80}{2 - 6i}$ **b.** $\frac{4 - 3i}{-1 + 2i}$

CONCEPTUAL UNDERSTANDING

EXAMPLE 5 Factor a Sum of Squares

How can you use complex numbers to factor the sum of two squares?

A. How can you factor the expression $x^2 + y^2$?

Rewrite $x^2 + y^2$ as a difference of two squares: $x^2 - (-y^2)$.

You can think of $(-y^2)$ as $(-1)(y^2)$.

Since $-1 = i^2$, $(-1)(y^2) = (i^2)(y^2) = (yi)^2$.

So $x^2 + y^2 = x^2 - (yi)^2$

$= (x + yi)(x - yi)$

The factors of $x^2 + y^2$ are $(x + yi)$ and $(x - yi)$.

How can $(-y^2)$ be a perfect square?

Factor as the difference of two squares.

STUDY TIP
The product of complex conjugates $(a + bi)$ and $(a - bi)$ will always be equal to $a^2 + b^2$, which is the sum of two squares.

B. How can you factor the expression $12x^2 + 3$?

$12x^2 + 3 = 3(4x^2 + 1)$ Factor out the GCF.

$= 3(4x^2 - i^2)$ Rewrite as a difference of squares.

$= 3(2x + i)(2x - i)$ Factor the difference of squares.

The factors of $12x^2 + 3$ are 3, $(2x + i)$, and $(2x - i)$.

Try It! **5.** Factor each expression.

a. $4x^2 + 25$ **b.** $8y^2 + 18$

EXAMPLE 6 Solve a Quadratic Equation With Complex Solutions

How can you solve $x^2 + 4 = 0$ using factoring?

$x^2 + 4 = 0$ Write the original equation.

$x^2 - (2i)^2 = 0$ Rewrite as a difference of squares.

$(x + 2i)(x - 2i) = 0$ Factor the difference of squares.

$x + 2i = 0 \quad x - 2i = 0$ Set each factor equal to 0.

$x = -2i \quad x = 2i$ Solve.

The solutions are $x = -2i$ and $x = 2i$.

LOOK FOR RELATIONSHIPS
In Example 1, you solved a similar problem by taking the square root of both sides. This example provides an alternative method that utilizes factoring.

Try It! **6.** Find the value(s) of x that will solve each equation.

a. $x^2 + 49 = 0$ **b.** $9x^2 + 25 = 0$

CONCEPT SUMMARY Complex Numbers and Operations

The imaginary unit i is the number whose square is equal to -1: $\sqrt{-1} = i$, so $i^2 = -1$.

Complex numbers are written in the form $a + bi$.

real numbers | imaginary unit

The four basic operations can be applied to complex numbers, such as $2 + 3i$ and $5 - i$.

ADDITION

Add as you would with binomials with like terms.

$(2 + 3i) + (5 - i) = 7 + 2i$

SUBTRACTION

Subtract as you would with binomials with like terms.

$(2 + 3i) - (5 - i) = -3 + 4i$

MULTIPLICATION

Distribute as you would with binomials.

$(2 + 3i)(5 - i) = 10 - 2i + 15i - 3i^2 = 13 + 13i$

DIVISION

Simplify so that the denominator is a real number. Multiply the numerator and denominator by the conjugate of the denominator.

$\frac{2 + 3i}{5 - i} = \frac{(2 + 3i)(5 + i)}{(5 - i)(5 + i)} = \frac{7 + 17i}{26} = \frac{7}{26} + \frac{17}{26}i$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you represent and operate on numbers that are not on the real number line?

2. **Vocabulary** How do you form the *complex conjugate* of a complex number $a + bi$?

3. **Error Analysis** Helena was asked to write the quotient $\frac{4}{3 - i}$ in the form $a + bi$. She began this way: $\frac{4}{3 - i} \times \frac{3 - i}{3 - i} = \frac{4(3 - i)}{3^2 + 1^2} = \frac{12 - 4i}{10}$. Explain the error Helena made.

4. **Look for Relationships** The quadratic equation $x^2 + 9 = 0$ has solutions $x = 3i$ and $x = -3i$. How many times will the graph of $f(x) = x^2 + 9$ cross the x-axis? Explain.

Do You KNOW HOW?

Write each of the following in the form $a + bi$.

5. $(2 + 5i) - (-6 + i)$

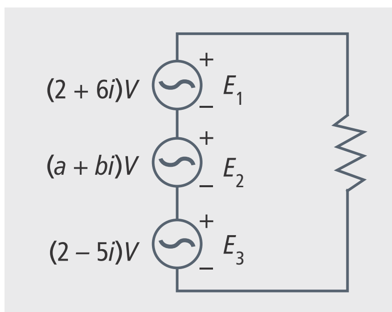
6. $(2i)(6 + 3i)$

Solve each equation.

7. $x^2 + 16 = 0$

8. $y^2 = -25$

9. **Model With Mathematics** The total source voltage in the circuit is $6 - 3i$ V. What is the voltage at the middle source?

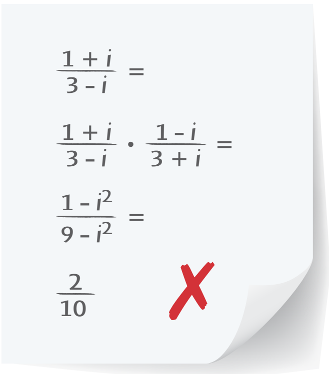


PRACTICE & PROBLEM SOLVING

UNDERSTAND

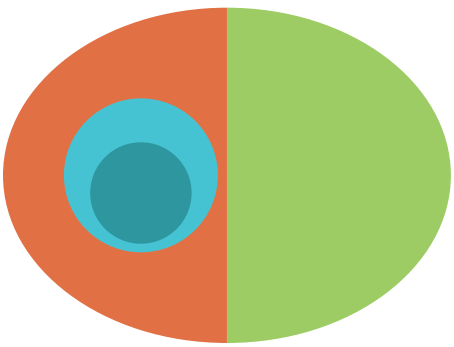
10. Construct Arguments Tamara says that raising the number i to any integer power results in either -1 or 1 as the result, since $i^2 = -1$. Do you agree with Tamara? Explain.

11. Error Analysis Describe and correct the error a student made when dividing complex numbers.



12. Higher Order Thinking Label the diagram with the following sets of numbers:

1. complex numbers
2. real numbers
3. imaginary numbers
4. integers
5. rational numbers



Include an example of each type of number in the diagram.

13. Generalize Write an explicit formula, in standard form, to find the quotient of two complex numbers. Use the numbers $a + bi$ and $c + di$.

PRACTICE

Use square roots to solve each equation over the complex numbers. SEE EXAMPLE 1

14. $x^2 = -5$

15. $x^2 = -0.01$

16. $x^2 = -18$

17. $x^2 = (-1)^2$

Add or subtract. Write the answer in the form $a + bi$. SEE EXAMPLE 2

18. $(3 - 2i) - (-9 + i)$

19. $(5 + 1.2i) + (-6 + 0.8i)$

20. $(2i) - (2i - 11)$

21. $13 + 2i - 4 - 8i$

22. $\frac{3-i}{4} - \frac{2+i}{3}$

23. $4.5i - 4.5 + 3.5i + 2.5$

Write each product in the form $a + bi$. SEE EXAMPLE 3

24. $(11i)(3i)$

25. $(3i)(5 - 4i)$

26. $(5 - 2i)(5 + 2i)$

27. $(8 + 3i)(8 + 3i)$

28. $\frac{1}{3}i(3 + 6i)$

29. $(-2i + 7)(7 + 2i)$

Write each quotient in the form $a + bi$.
SEE EXAMPLE 4

30. $\frac{12}{1-i}$

31. $\frac{5}{6+2i}$

32. $\frac{6+12i}{3i}$

33. $\frac{4-4i}{1+3i}$

Factor the sums of two squares. SEE EXAMPLE 5

34. $4x^2 + 49$

35. $x^2 + 1$

36. $36 + 100a^2$

37. $18y^2 + 8$

38. $\frac{1}{4}b^2 + 25$

39. $x^2 + y^2$

Solve each equation. SEE EXAMPLE 6

40. $x^2 + 81 = 0$

41. $25x^2 + 9 = 0$

42. $x^2 = -16$

43. $4 + 49y^2 = 0$

44. $y^2 + 1 = 0$

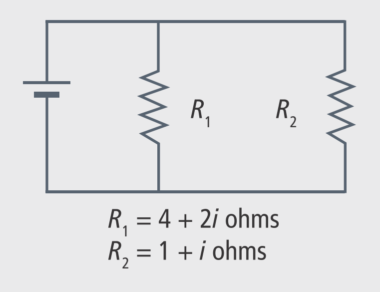
45. $x^2 + \frac{1}{4} = 0$

PRACTICE & PROBLEM SOLVING

Practice Tutorial
Mixed Review Available Online

APPLY

46. Model With Mathematics The two resistors shown in the circuit are referred to as *in parallel.* The total resistance of the resistors is given by the formula $\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2}$.

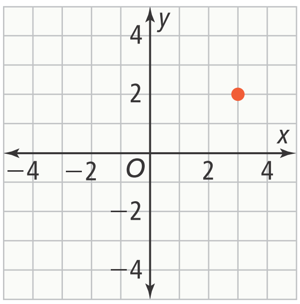


a. Find the total resistance. Write your answer in the form $a + bi$.

b. Show that the total resistance is equivalent to the expression $\frac{R_1 R_2}{R_1 + R_2}$.

c. Change the value of R_2 so that the total resistance is a real number. Explain how you chose the value.

47. Use Structure The complex number $a + bi$ can be represented on a coordinate plane as the point (a, b). You can use multiplication by i to rotate a point about the origin in the coordinate plane.



a. Write the point (x, y) on the graph as the complex number $x + yi$.

b. Multiply the complex number by i. Interpret the new value as a new point in the plane.

c. Repeat the steps above for two other points. How does multiplication by i rotate a point?

ASSESSMENT PRACTICE

48. Select all powers of i that are real numbers.

N-CN.1.1

☐ **A.** i
☐ **B.** i^2
☐ **C.** i^3
☐ **D.** i^4
☐ **E.** i^5
☐ **F.** i^6

49. SAT/ACT Which of the following is a solution to the equation $3x^2 = -12$?

Ⓐ $-4i$

Ⓑ $-2i$

Ⓒ -2

Ⓓ 2

Ⓔ $4i$

50. Performance Task Abby wants to write the square root of i in the form $a + bi$. She begins by writing the equation $\sqrt{i} = a + bi$.

Part A Square both sides of the equation. Then use the fact that the real part and imaginary part on each side of the equation are equal to write a system of equations involving the variables a and b.

Part B Solve the system to find b. Then find a.

Part C List the possible solutions for a and b.

Part D Square each of the possible solutions. What are the two square roots of i?